Problem Set 12

R is a commutative ring (with 1, as always in this course) throughout, and K denotes an algebraically closed field containing k.

1. (a) Prove that C[z] is integrally closed (in its fraction field C(z)), i.e. normal. [Hint: suppose you had a \( \frac{P}{Q} \in C(z) \), P and Q relatively prime in C[z], integral over C[z].]
(b) Prove the same result for any UFD.

2. In lecture I described normalizing the curve \( y^2 = x^3 - x^2 \) with a nodal singularity at \((0,0)\) by introducing the function \( z = \frac{y}{x} \). If \( R = \frac{C[x,y]}{(y^2 - x^3 + x^2)} \), then let \( S = \frac{C[x,u,v]}{(y^2 - x^3 + x^2, z^2 - x + 1, z^3 + z - y)} \) with the natural map \( \varphi: R \to S \). Show
(a) that \( S \cong C[z] \) (the coordinate ring of a complex line!).
(b) What geometric map does \( \varphi \) correspond to “pulling back functions” along? Use this to argue that \( \varphi \) is injective (or prove this by some other means).
(c) Show (e.g. using Chinese remainder) that \( T = \frac{C[x,y,z]}{(y^2 - x^3 + x^2, z^2 - x + 1, z^3 + z - y)} \) is not a domain, so this can’t be inside \( R \)’s field of fractions.
(d) Use problem (1) to show that \( S \) is the integral closure of \( R \) in its fraction field. [Hint: don’t make this problem hard. It’s all very simple calculations or trivial arguments.]

3. Let \( R \) be a Noetherian local ring with maximal ideal \( m \). If the ideal \( m / m^2 \) in \( R / m^2 \) is generated by \( \{a_1 + m_2, \ldots, a_n + m_2\} \), show that the ideal \( m \) is generated in \( R \) by \( \{a_1, \ldots, a_n\} \). [Hint: use a version of Krull’s intersection theorem.]

4. Let \( S \) be an integral extension ring of \( R \) and suppose \( R \) and \( S \) are domains. Show that \( S \) is a field if and only if \( R \) is a field.

5. If \( V_1 \supseteq V_2 \supseteq \cdots \) is a descending chain of k-varieties in \( K^n \), show that \( V_m = V_{m+1} = \cdots \) for some \( m \). [Hint: you may wish to use the 1-to-1 inclusion-reversing correspondence between radical ideals and varieties, and the Hilbert basis theorem.]

6. If \( I_1, \ldots, I_m \) are ideals of \( k[x_1, \ldots, x_n] \), show that \( V(I_1 \cap \cdots \cap I_m) = V(I_1) \cup \cdots \cup V(I_m) \) and \( V(I_1 + \cdots + I_m) = V(I_1) \cap \cdots \cap V(I_m) \). [Hint: it may help to use properties of Rad( ).]

7. A k-variety \( V \) in \( K^n \) is irreducible provided that whenever \( V = W_1 \cup W_2 \) with each \( W_i \) a k-variety in \( K^n \), either \( V = W_1 \) or \( V = W_2 \).
(a) Prove that \( V \) is irreducible if and only if \( J(V) \) is a prime ideal in \( k[x_1, \ldots, x_n] \).
(b) Let \( K = \mathbb{C} \) and \( S = \{x_1^2 - 2x_2^3\} \). Show that \( V(S) \) is irreducible as a \( \mathbb{C} \)-variety but not as an \( \mathbb{R} \)-variety.