Problem Set 12

Some of the problems use material we'll cover on Monday or Wednesday. R is a commutative ring (with 1, as always in this course). Below K denotes an algebraic closure of k.

(1) (a) Prove that $\mathbb{C}[z]$ is integrally closed (in its fraction field $\mathbb{C}(z)$), i.e. "normal". [Hint: suppose you had a $\frac{P}{Q} \in \mathbb{C}(z)$, P and Q relatively prime in $\mathbb{C}[z]$, integral over $\mathbb{C}[z]$.] (b) Prove the same result for any UFD.

(2) In class I described "normalizing" the curve $y^2 = x^3 - x^2$ with a nodal singularity at $(0,0)$ by introducing the function "$z = y/x$". If $R = \frac{\mathbb{C}[x,y]}{(y^2-x^3+x^2)}$, then let $S = \frac{\mathbb{C}[x,y,z]}{(y^2-x^3+x^2, z^2-x+1, z^3+z-y)}$ with the natural map $\phi : R \to S$. Show (a) that $S \cong \mathbb{C}[z]$ (the coordinate ring of a complex line!). (b) What geometric map does ϕ correspond to "pulling back functions" along? Use this to argue that ϕ is injective (or prove this by some other means). (c) Show (e.g. using Chinese remainder) that if you use $S' = \frac{\mathbb{C}[x,y]}{(y^2-x^3+x^2, z^2-x+1)}$ as I suggested in class, you don't get a domain, so this can't be inside R's field of fractions. (d) Use problem (1) to show that S is the integral closure of R in its fraction field. [Hint: don't make this problem hard. It's all very simple calculations or trivial arguments.]

(3) Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$. If the ideal $\mathfrak{m}/\mathfrak{m}^2$ in $R/\mathfrak{m}^2$ is generated by $\{a_1 + \mathfrak{m}^2, \ldots, a_n + \mathfrak{m}^2\}$, show that the ideal $\mathfrak{m}$ is generated in R by $\{a_1, \ldots, a_n\}$. [Hint: use the result I called the "classical" Krull's intersection theorem.]

(4) Let S be an integral extension ring of R and suppose R and S are domains. Show that S is a field if and only if R is a field.

(5) Show that every affine k-variety in K^n is of the form $V(\mathcal{S})$ where $\mathcal{S}$ is a finite subset of $k[x_1, \ldots, x_n]$. [Hint: Use the 1-to-1 inclusion-reversing correspondence between radical ideals and varieties (will do in class), and the Hilbert basis theorem.]

(6) If $V_1 \supset V_2 \supset \cdots$ is a descending chain of k-varieties in K^n, then $V_m = V_{m-1} = \cdots$ for some m. [Hint: same as for problem (5).]

(7) If $I_1, \ldots, I_m$ are ideals of $k[x_1, \ldots, x_n]$, then $V(I_1 \cap \cdots \cap I_m) = V(I_1) \cup \cdots \cup V(I_m)$ and $V(I_1 + \cdots + I_m) = V(I_1) \cap \cdots \cap V(I_m)$. [Hint: it may help to use properties of $Rad(\cdot)$.]

(8) A k-variety V in K^n is *irreducible* provided that whenever $V = W_1 \cup W_2$ with each W_i a k-variety in K^n, either $V = W_1$ or $V = W_2$.

(a) Prove that V is irreducible if and only if $J(V)$ is a prime ideal in $k[x_1, \ldots, x_n]$.

(b) Let $K = \mathbb{C}$ and $\mathcal{S} = \{x_1^2 - 2x_2^2\}$. Show that $V(\mathcal{S})$ is irreducible as a $\mathbb{Q}$-variety but not as an $\mathbb{R}$-variety.