PROBLEM SET 12

R is a commutative ring (with 1, as always in this course) throughout, and K denotes an algebraically closed field containing k.

(1) Prove that C[z] is integrally closed (in its fraction field C(z)), i.e. normal. [Hint: suppose you had $P / Q \in C(z)$, P and Q relatively prime in C[z], integral over C[z].]

(2) In lecture I described normalizing the curve $y^2 = x^3 - x^2$ with a nodal singularity at (0,0) by introducing the function $z = \frac{y}{x}$. If $R = \frac{C[x,y]}{(y^2-x^3+x^2)}$, then let $S = \frac{C[x,y,z]}{(y^2-x^3+x^2,z^2-x+1,z^3+y)}$ with the natural map $\phi: R \rightarrow S$. Show
   (a) that $S \cong C[z]$ (the coordinate ring of a complex line!).
   (b) What geometric map does $\phi$ correspond to “pulling back functions” along? Use this to argue that $\phi$ is injective (or prove this by some other means).
   (c) Show (e.g. using Chinese remainder) that $T = \frac{C[x,y,z]}{(y^2-x^3+x^2,z^2-x-1)}$ is not a domain, so this can’t be inside R’s field of fractions.
   (d) Use problem (1) to show that S is the integral closure of R in its fraction field.
   [Hint: don’t make this problem hard. It’s all very simple calculations or trivial arguments.]

(3) Let R be a Noetherian local ring with maximal ideal m. If the ideal m/m² in R/m² is generated by $\{a_1 + m^2, \ldots, a_n + m^2\}$, show that the ideal m is generated in R by $\{a_1, \ldots, a_n\}$. [Hint: use Nakayama’s Lemma (i) $\implies$ (iv).]

(4) Let S be an integral extension ring of R and suppose R and S are domains. Show that S is a field if and only if R is a field.

(5) If $V_1 \supset V_2 \supset \cdots$ is a descending chain of k-varieties in $K^n$, show that $V_m = V_{m+1} = \cdots$ for some m. [Hint: you may wish to use the 1-to-1 inclusion-reversing correspondence between radical ideals and varieties, and the Hilbert basis theorem.]

(6) If $I_1, \ldots, I_m$ are ideals of $k[x_1, \ldots, x_n]$, show that $V(I_1 \cap \cdots \cap I_m) = V(I_1) \cup \cdots \cup V(I_m)$ and $V(I_1 + \cdots + I_m) = V(I_1) \cap \cdots \cap V(I_m)$. [Hint: it may help to use properties of Rad(·).]

(7) A k-variet $V$ in $K^n$ is irreducible provided that whenever $V = W_1 \cup W_2$ with each $W_i$ a k-variet in $K^n$, either $V = W_1$ or $V = W_2$.
   (a) Prove that $V$ is irreducible if and only if $J(V)$ is a prime ideal in $k[x_1, \ldots, x_n]$.
   (b) Let $K = C$ and $S = \{x_1^2 - 2x_2^2\}$. Show that $V(S)$ is irreducible as a Q-variet but not as an R-variet.