Problem Set 2

Hand in all.

(1) [Jacobson p. 229 #2] Construct a splitting field over \( \mathbb{Q} \) of \( x^5 - 2 \). Find its degree over \( \mathbb{Q} \).

(2) [Jacobson p. 229 #3] Determine a splitting field over \( \mathbb{Z}_p \) of \( x^{p^e} - 1 \), \( e \in \mathbb{N} \).

(3) Find splitting field extensions for \( x^3 - 5 \) over \( \mathbb{Z}_7, \mathbb{Z}_{11} \) and \( \mathbb{Z}_{13} \).

(4) Show that an algebraically closed field must be infinite.

(5) Suppose that \( K(\alpha)/K \) is a simple extension and that \( \alpha \) is transcendental over \( K \).
Show that \( K(\alpha) \) is not algebraically closed.

(6) [Jacobson p. 234 #2] Let \( f(x) \) be irreducible in \( F[x] \), where \( F \) is of characteristic \( p \).
Show that \( f(x) \) can be written as \( g(x^{p^e}) \), where \( g(x) \) is irreducible and separable.
Use this to show that every root of \( f(x) \) has the same multiplicity \( p^e \) (in a splitting field).

(7) [Jacobson p. 234 #4] Let \( F \) be imperfect of characteristic \( p \). Show that \( x^{p^e} - a \) is irreducible if \( a \not\in F^p \) and \( e = 0, 1, 2, \ldots \).

(8) Let \( p \) be a prime number. By factorizing \( x^{p-1} - 1 \) over \( \mathbb{Z}_p \), prove Wilson’s theorem:
i.e., show that \( (p-1)! \equiv -1 \pmod{p} \). (Compare #5 on Problem Set 8 from Algebra I.)

(9) Let \( K \) be a field of positive characteristic.
   (i) Show that \( K \) is perfect if and only if the Frobenius homomorphism is an automorphism.
   (ii) If \( L/K \) is a totally inseparable extension (i.e. every element of \( L \setminus K \) is inseparable), show that the minimal polynomial of any element of \( L \) over \( K \) is of the form \( x^{p^n} - \alpha \), where \( \alpha \in K \).