Problem Set 3

[J]=Jacobson

(1) Suppose that L/K is a finite normal extension and that f is an irreducible polynomial in $K[x]$. Suppose that g and h are irreducible monic factors of f in $L[x]$. Show that there is an automorphism σ of L which fixes K such that $\sigma(g) = h$.

(2) Suppose that L/K is a Galois extension, $G = Aut(L/K)$, and $\alpha \in L$. Show that $L = K(\alpha)$ if and only if the images of α under the elements of G are all distinct.

(3) Suppose that L/K is a Galois extension with Galois group $G = \{\sigma_1, \ldots, \sigma_n\}$. Show that $(\beta_1, \ldots, \beta_n)$ is a basis for L over K if and only if $\det(\sigma_i(\beta_j)) \neq 0$.

(4) Find the Galois group of $f = x^4 - 2$ over $\mathbb{Q}$, $\mathbb{Z}_3$, and $\mathbb{Z}_7$.

(5) Given a finite group G show that there exists a Galois extension L/K such that $Aut(L/K) \cong G$.

(6) Let L/K be an extension of degree 2, that $L^2 = L$, char$K \neq 2$, and that every polynomial of odd degree in $K[x]$ has a root in K. Given $f \in K[x]$, let M/L be a splitting field extension for f over L; and put $G := Aut(M/K)$, with subgroup $H := Aut(M/L)$.
(i) Show that $|G| = 2^n$. [Hint: consider the fixed field of a Sylow 2-subgroup of G.]
(ii) Show that if $n > 1$ then there is an irreducible quadratic in $L[x]$. [Hint: consider a subgroup of index 2 in H.]
(iii) Show that L is algebraically closed.
(iv) Apply your result to $\mathbb{C}/\mathbb{R}$ to obtain an algebraic proof that $\mathbb{C}$ is algebraically closed.

(7) [J] p. 243 #1

(8) [J] p. 243 #3

(9) [J] p. 243 #7

(10) [J] p. 243 #9