Problem Set 3

(1) [Jacobson p. 229 #2] Construct a splitting field over \( \mathbb{Q} \) of \( x^5 - 2 \). Find its degree over \( \mathbb{Q} \).

(2) [Jacobson p. 229 #3] Determine a splitting field over \( \mathbb{Z}_p \) of \( x^{p^e} - 1, e \in \mathbb{N} \).

(3) Find splitting field extensions for \( x^3 - 5 \) over \( \mathbb{Z}_7, \mathbb{Z}_{11} \) and \( \mathbb{Z}_{13} \).

(4) Show that an algebraically closed field must be infinite.

(5) Suppose that \( K(\alpha)/K \) is a simple extension and that \( \alpha \) is transcendental over \( K \). Show that \( K(\alpha) \) is not algebraically closed.

(6) Let \( p \) be a prime number. By factorizing \( x^{p-1} - 1 \) over \( \mathbb{Z}_p \), prove Wilson’s theorem:
   i.e., show that \( (p-1)! \equiv -1 \pmod{p} \). (I know it’s a retread from Algebra I, but this gives a quicker proof and new perspective.)

(7) [Jacobson p. 234 #2] Let \( f(x) \) be irreducible in \( F[x] \), where \( F \) is of characteristic \( p \). Show that \( f(x) \) can be written as \( g(x^{p^e}) \), where \( g(x) \) is irreducible and separable. Use this to show that every root of \( f(x) \) has the same multiplicity \( p^e \) (in a splitting field). [Hint: use I.E.6 and I.E.8 from the Algebra II notes.]