PROBLEM SET 4

Hand in all.

(1) Given a finite group G, show that there exists a Galois extension L/K such that $Aut(L/K) \cong G$.
(2) [Jacobson, p. 243 #9] Show that $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, u)$ where $u^2 = (9 - 5\sqrt{3})(2 - \sqrt{2})$, is normal. Determine $\mathrm{Gal}(E/\mathbb{Q})$.
(3) Factorize $x^{p^p} - x$ over $\mathbb{Z}_p$.
(4) Let $p < q$ be primes, $p \nmid q - 1$. Show that there is an extension $L/\mathbb{Z}_q$ which is a splitting field extension for each of the polynomials $x^p - a$ ($a \in \mathbb{Z}_q^*$).
(5) Suppose that L/K is finite and separable and M/L is finite simple. Show that M/K is simple.
(6) Show that the simple transcendental extension $K(t)/K$ has infinitely many intermediate fields.
(7) [Jacobson, p. 151 #19] Prove that if $\varphi(n)$ is the Euler phi-function, then $\varphi(n) = \sum_{d|n} \mu(\frac{n}{d})d$.
(8) [Jacobson, p. 295 #2] Find a primitive element for a splitting field over $\mathbb{Q}$ of $x^5 - 2$.