Problem Set 6

[J] = Jacobson. (Note: problem 9 is based on Wednesday’s lecture.)

1. [J] p. 276 #2
2. [J] p. 277 #3
3. [J] p. 277 #4 and p. 243 #2
4. [J] p. 287 #1
5. Find the Galois groups of \( x^4 + 1 \) and \( x^5 + 1 \) over \( \mathbb{Q} \).
6. Suppose that \( p \) is prime and doesn’t divide \( m \), and let \( \varepsilon \) be a primitive \( m^{th} \) root of 1 over \( \mathbb{Z}_p \). Show that \( [\mathbb{Z}_p(\varepsilon) : \mathbb{Z}_p] = k \), where \( k \) is the order of \( \bar{\varepsilon} \) in \( \mathbb{Z}_m^* \). Show that the cyclotomic polynomial \( \lambda_m \) is irreducible over \( \mathbb{Z}_p \) if and only if \( \mathbb{Z}_m^* \) is a cyclic group generated by \( \bar{p} \). When is \( \lambda_4 \) irreducible over \( \mathbb{Z}_p \)? How about \( \lambda_8 \)?
7. Using Thm 4.2.1 in [J], determine whether \( \lambda_{18} \) is irreducible over \( \mathbb{Z}_{23}, \mathbb{Z}_{43}, \mathbb{Z}_{73} \).
8. Suppose \( L/K \) is an extension, and that \( L \) is finitely generated over \( K \). Show that the field \( K_a \) of elements of \( L \) which are algebraic over \( K \) is finitely generated over \( K \).
9. Show that the primitive \( n^{th} \) roots of unity over \( \mathbb{Q} \) form a normal basis for the splitting field of \( x^n - 1 \) over \( \mathbb{Q} \) if and only if \( n \) has no repeated prime factors.