# Problem Set 6

[J]=Jacobson. (Note: problem 9 is based on Wednesday's lecture.)

(1) [J] p. 276 #2
(2) [J] p. 277 #3
(3) [J] p. 277 #4 and p. 243 #2
(4) [J] p. 287 #1
(5) Find the Galois groups of $x^4 + 1$ and $x^5 + 1$ over $\mathbb{Q}$.
(6) Suppose that $p$ is prime and doesn't divide $m$, and let $\varepsilon$ be a primitive $m^{\text{th}}$ root of 1 over $\mathbb{Z}_p$. Show that $[\mathbb{Z}_p(\varepsilon) : \mathbb{Z}_p] = k$, where $k$ is the order of $\bar{p}$ in $\mathbb{Z}_m^*$. Show that the cyclotomic polynomial $\lambda_m$ is irreducible over $\mathbb{Z}_p$ if and only if $\mathbb{Z}_m^*$ is a cyclic group generated by $\bar{p}$. When is $\lambda_4$ irreducible over $\mathbb{Z}_p$? How about $\lambda_8$?
(7) Using Thm 4.2.1 in [J], determine whether $\lambda_{18}$ is irreducible over $\mathbb{Z}_{23}$, $\mathbb{Z}_{43}$, $\mathbb{Z}_{73}$.
(8) Suppose $L/K$ is an extension, and that $L$ is finitely generated over $K$. Show that the field $K_a$ of elements of $L$ which are algebraic over $K$ is finitely generated over $K$.
(9) Show that the primitive $n^{\text{th}}$ roots of unity over $\mathbb{Q}$ form a normal basis for the splitting field of $x^n - 1$ over $\mathbb{Q}$ if and only if $n$ has no repeated prime factors.