PROBLEM SET 7

(1) Suppose that $f = x^n + px + q$. Show that $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} = 0$, $\lambda_{n-1} = -(n-1)p$, $\lambda_n = -nq$, $\lambda_{n+1} = \cdots = \lambda_2 = (n-1)p^2$. Hence show the discriminant $\Delta$ of $f$ is $\varepsilon_n n^n q^{n-1} + \varepsilon_{n-1} (n-1)^{n-1} p^n$, where $\varepsilon_n = 1$ if $n \equiv 0, 1$ and $\varepsilon_n = -1$ if $n \equiv 2, 3$. [Hint: use Newton’s identities on p. 140 of Jacobson.]

(2) Using our methods for dealing with quartics (and problem (1)), determine Galois groups (over $\mathbb{Q}$) of
(a) $x^4 + 8x - 12$,
(b) $x^4 + 1$,
(c) $x^4 + x^3 + x^2 + x + 1$,
(d) $x^4 - 2$.

(3) Using what we’ve learned about cyclotomic polynomials $\Phi_m$, determine Galois groups (over $\mathbb{Q}$) of $x^4 + 1$ and $x^5 + 1$. [Hint: you may use Theorem 4.21 of Jacobson. I’m aware that this repeats one of the polynomials in (2).]

(4) Suppose that $p$ is prime and does’t divide $m$, and let $\varepsilon$ be a primitive $m^{th}$ root of 1 over $\mathbb{Z}_p$. Show that $[\mathbb{Z}_p(\varepsilon):\mathbb{Z}_p] = k$, where $k$ is the order of $\varepsilon$ in $\mathbb{Z}_m^*$. Show that the cyclotomic polynomial $\Phi_m$ is irreducible over $\mathbb{Z}_p$ if and only if $\mathbb{Z}_m^*$ is a cyclic group generated by $\varepsilon$. When is $\Phi_4$ irreducible over $\mathbb{Z}_p$? How about $\Phi_8$?

(5) Determine whether $\Phi_{18}$ is irreducible over $\mathbb{Z}_{23}$, $\mathbb{Z}_{43}$, and $\mathbb{Z}_{73}$. (You may want to look at I.L.22 and the comments after it.)

(6) Show that the primitive $n^{th}$ roots of 1 over $\mathbb{Q}$ form a normal basis for the splitting field of $x^n - 1$ over $\mathbb{Q}$ if and only if $n$ has no repeated prime factors.

(7) [Jacobson, p. 287 #1] Show that $\sin(u)$ is transcendental for all algebraic $u \neq 0$.

(8) Suppose $L/K$ is an extension, and that $L$ is finitely generated over $K$. Show that the field $K_a$ of elements of $L$ which are algebraic over $K$ is f.g. over $K$. 

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