Problem Set 7

[J]=Jacobson. (You need to do these over break, but you don't need to write them up and turn them in.) Problems 6-9 go together, and prove a special case of the Kronecker-Weber theorem (the statement that any abelian extension of $\mathbb{Q}$, i.e. Galois extension with abelian automorphism group, is a subfield of a cyclotomic field).

(1) [J] p. 300 #1
(2) [J] p. 305 #3
(3) [J] p. 305 #4
(4) [J] p. 305 #8
(5) The polynomial $f = x^7+9x^2+7$ is irreducible over $\mathbb{Q}$. By reducing mod 7, show that its Galois group contains a 4-cycle. Use this together with our theorem that any soluble Galois group would have to be contained in W_7, to show that f is not solvable by radicals.
(6) Let p be an odd prime. Show that if $r \in \mathbb{Z}$ then $\sum_{0\leq s\leq p} \zeta_p^{rs}$ equals p if $r \underset{(p)}{\equiv} 0$ and equals 0 otherwise.
(7) Let τ be the Gauss sum $\sum_{0\leq n<p} \zeta_p^{n^2}$. Show that $\tau\bar{\tau} = p$. Show also that τ is real if -1 is a square mod p, and otherwise purely imaginary.
(8) Let $F = \mathbb{Q}(\zeta_p)$. Show that F has a unique subfield K which is quadratic over $\mathbb{Q}$, and that $K = \mathbb{Q}(\sqrt{\varepsilon p})$ where $\varepsilon = (-1)^{(p-1)/2}$.
(9) Show that $\mathbb{Q}(\zeta_M) \subset \mathbb{Q}(\zeta_N)$ if $M|N$. Deduce that if $0 \neq m \in \mathbb{Z}$ then $\mathbb{Q}(\sqrt{m})$ is a subfield of $\mathbb{Q}(\zeta_{4|m|})$.