PROBLEM SET 8

(1) [Jacobson p. 100 #7] Let \( m \) and \( n \) be non-zero integers and let \( R \) be the subset of \( M_2(\mathbb{C}) \) consisting of the matrices of the form

\[
\begin{pmatrix}
    a + b\sqrt{m} & c + d\sqrt{m} \\
    n(c - d\sqrt{m}) & a - b\sqrt{m}
\end{pmatrix}
\]

where \( a, b, c, d \in \mathbb{Q} \). Show that \( R \) is a subring of \( M_2(\mathbb{C}) \) and that \( R \) is a division ring if and only if the rational numbers \( x, y, z, t \) satisfying the equation \( x^2 - my^2 - nz^2 + mnt^2 = 0 \) are \( x = y = z = t = 0 \). Give a choice of \( m, n \) for which \( R \) is a division ring and a choice of \( m, n \) for which \( R \) is not a division ring. [N.B. These rings are called “rational quaternion algebras”.

(2) [Jacobson, p. 300 #1] Show that if \( E \) is a finite field and \( F \) is a subfield, so that \( E/F \) is a cyclic extension, then the norm homomorphism \( N_{E/F} \) of \( E^* \) is surjective on \( F^* \).

(3) In this problem you will prove a special case of the Kronecker-Weber theorem, that every abelian extension of \( \mathbb{Q} \) is a subfield of a cyclotomic field. The first 3 parts come from [Jacobson, pp. 276-277].

(a) Suppose \( f(x) \in K[x] \) (of degree \( n \)) has \( n \) distinct roots \( r_i \) in a splitting field.

Show that the discriminant \( \Delta \) is equal to \((-1)^{n(n-1)/2} \prod_{i=1}^{n} f'(r_i)\).

(b) Let \( p \) be an odd prime. By differentiating \( x^p - 1 = (x - 1)\Phi_p(x) \), show that the discriminant of \( \Phi_p \) is \((-1)^{p(p-1)/2}p^{p-2}\).

(c) Show that \( \mathbb{Q}(\zeta_p) \) has a unique subfield \( E \) with \( [E:Q] = 2 \), which is real or not depending on whether \( p \) has the form \( 4n + 1 \) or \( 4n + 3 \).

(d) Prove that for each \( m \in \mathbb{Z}\setminus\{0\} \), \( \mathbb{Q}(\sqrt{m}) \) is a subfield of \( \mathbb{Q}(\zeta_{4|m|}) \). [Hint: don’t forget that \( \mathbb{Q}(\zeta_M) \subset \mathbb{Q}(\zeta_N) \) if \( M \mid N \).]