Hand in all.

(1) [Jacobson, p. 404 #1]
(2) Show that $U(1)$ is a real form of $\mathbb{C}^*$, by following the hint in II.B.3(C).
(3) What are the semisimple $\mathbb{Z}$-modules?
(4) Show that a (left) $R$-module $M$ has a composition series iff it is Noetherian and Artinian, cf. the remarks just before III.A.4.
(5) Show that the center of a semisimple ring $R$ is a finite direct product of fields. Show that, in the case $R \cong M_n(D)$ ($D$ a division ring), it is a field.
(6) Let $M$ be a finitely generated (left) $R$-module and $E = \text{End}_R(M)$. Show that if $R$ is semisimple, then so is $E$.
(7) Let $R$ be an $n^2$-dimensional algebra over a field $k$. Show that $R \cong M_n(k)$ (as $k$-algebras) if and only if $R$ is simple$^1$ and has an element whose minimal polynomial over $k$ has the form $(x - a_1) \cdots (x - a_n)$ where $a_i \in k$. [Hint: for the “if” part, observe that, for the given $r$, $k[r] \cong k \times \cdots \times k$ ($n$ copies) and show that $R$ has a composition series of length $\geq n$.]
(8) Let $R$ be a central simple algebra over $k$. Show that $R$ is isomorphic to a matrix algebra over over $k$ if and only if $R$ has a nonzero right ideal $I$ with $(\dim_k I)^2 \leq \dim_k R$. $^2$

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$^1$Here I mean simple in the weaker ($k$-algebra) sense, that $R$ has no nontrivial proper 2-sided ideals.

$^2$I deleted part (i) of this problem, since I forgot and proved it in III.C.7.