Problem Set 9

This one is hopefully on the light side. We'll get to Maschke's theorem on Wednesday.

(1) What are the semisimple $\mathbb{Z}$-modules?

(2) Show that the center of a semisimple ring R is a finite direct product of fields. Show that, in the case $R \cong M_n(D)$ (D a division ring), it is a field.

(3) Let M be a finitely generated (left) R-module and $E = End_R(M)$. Show that if R is semisimple, then so is E.

(4) (i) Prove that a simple ring[1] R which is finite-dimensional over its center is also semisimple, hence isomorphic to some $M_n(D)$, D a division algebra over k (the center, now clearly a field). [Hint: one way to proceed is to establish the existence of a minimal left ideal I, then consider the sum of all ideals which are isomorphic to I as left R-modules.]

(ii) Show that R is isomorphic to a matrix algebra over its center if and only if R has a nonzero right ideal I with $(\dim_k I)^2 \leq \dim_k R$.

(5) Let R be an n^2-dimensional algebra over a field k. Show that $R \cong M_n(k)$ (as k-algebras) if and only if R is simple and has an element whose minimal polynomial over k has the form $(x - a_1) \cdot \cdots \cdot (x - a_n)$ where $a_i \in k$. [Hint: for the "if" part, observe that, for the given r, $k[r] \cong k \times \cdots \times k$ (n copies) and show that[2] ${}_R R$ has a composition series of length $\geq n$.]

(6) Let V be a $k[G]$-module and H ne a subgroup in G of finite index not divisible by $\operatorname{char}(k)$. Modify the proof of Maschke's theorem to show the following: if V is semisimple as a $k[H]$-module, then V is semisimple as a $k[G]$-module.

[1] by definition, this means a ring with no proper nontrivial 2-sided ideals
[2] the notation ${}_R R$ means "R viewed as a left R-module"