HW #10

Solutions

1. Fix $a \neq 0$ and consider the descending chain $(a) \supset (a^2) \supset (a^3) \ldots$. Since the ring is Artinian, this must stabilize, and we have $(a^n) = (a^{n+1})$ for some n. Then there exists $r \in R$ such that $a^n = ra^{n+1}$, and canceling (permissible since R is a domain) we find $1 = ra$, so r is an inverse for a. Since the same argument shows every element has an inverse, the ring is a field.

2. Let R be a left Noetherian ring and $\phi : R \to S$ a ring homomorphism. let $\{I_i\}$ be an ascending chain of left ideals in S. By the correpondence theorem, these pull back to an ascending chain of left ideals $\{\phi^{-1}(I_j)\}$ containing $\ker \phi$ in R. Since these stabilize, so must their images $\phi(\phi^{-1}(I_j) = I_j$. An analogous argument applies to left Artinian rings.

3. a. Suppose $r/s \in S^{-1} \operatorname{Rad} I$. Then $r^n \in I$ for some n, so $(r/s)^n \in S^{-1}I$ and $r/s \in \operatorname{Rad}(S^{-1}I)$. If $r/s \in \operatorname{Rad}(S^{-1}I)$, then $r^n/s^n \in S^{-1}I$ for some n, and $r^n \in I$, so $r \in \operatorname{Rad} I$ and $r/s \in\in S^{-1} \operatorname{Rad} I$. The desired equality follows.

   b. Suppose R is Noetherian, and consider an ascending chain of ideals $\{I_j\}$ in $S^{-1}R$. The ideals are images of ideals K_j in R not meeting S under localization. Since the K_j must stabilize, so must their images.

4. Suppose R is a local ring and there exist $r, s \in R$ with $r + s = 1$ such that neither r nor s is a unit. Then since (r) and (s) are each contained in the unique maximal ideal M, $R = (r) + (s) \subset M$, a contradiction.

   Suppose R is a ring in which for every $r, s \in R$, $r + s = 1$ implies either r or s is a unit. Suppose there are (at least) two maximal ideals M_1 and M_2 in the ring. Then $M_1 + M_2 = I$ for some ideal I. If I is a proper ideal, then at least one of M_i is not maximal, so $I = R$, and there exist nonunits $m_1 \in M_1$ and $m_2 \in M_2$ with $m_1 + m_2 = 1$, a contradiction. So there is at most one maximal ideal, and the ring is local.

5. The localization of $\mathbb{Z}$ at $p\mathbb{Z}$ is just all "fractions" a/b for $a \in \mathbb{Z}$ and $b \in \mathbb{Z} - p\mathbb{Z}$. From work in class, we know this is a local ring with single maximal ideal $p(S^{-1}R)$. Quotienting by this ideal gives a field isomorphic to $\mathbb{Z}/p\mathbb{Z}$. To see this, consider the image of some r/s in the local ring under the quotient map. Note that we may clearly reduce the numerator by multiples of p, and that we may reduce the denominator by multiples of p by noting subtracting $r/(s-p)$ gives an element of the maximal ideal in the local ring. We then get all fractions a/b with a and b reduced mod p. Next, note that if $rs = 1$ mod p, then $1/r = s$ when take the quotient of the local ring, so we can clear denominators to get $\mathbb{Z}/p\mathbb{Z}$.

6. a. Note that any one-dimensional representation is irreducible. We have two: the trivial representation and the representation given by $g \to (-1)^{\operatorname{sgn} g}$. There is also the two-dimensional standard representation. If S_3 acts by permutation on the basis vectors e_1, e_2, e_3, it descends to a representation ϕ on $\mathbb{Q}^3/\langle e_1 + e_2 + e_3 \rangle$. This has basis $\bar{e}_1$ and $\bar{e}_2$. If $a\bar{e}_1 + b\bar{e}_2$ is a nonzero vector in this space, then applying $\phi((12))$ gives a vector $b\bar{e}_1 + a\bar{e}_2$, and these two vectors span the space if $a \neq b$. Otherwise, applying $\phi((13))$ to the vector produces a basis. So there is no invariant subspace, and this representation is irreducible.

   The 1-dimensional representations must correspond to 1×1 matrices over $\mathbb{Q}$ in the Wedderburn decomposition, but we need to show the remaining 4 dimensional component is actually $M_2(\mathbb{Q})$ and not $M_1(D)$ for some 4-dimensional division algebra. If the representations we have found are ρ_i, it suffices to show the map $\mathbb{Q}[G] \to \mathbb{Q} \times \mathbb{Q} \times M_2(\mathbb{Q})$ given by (ρ_1, ρ_2, ρ_3) is injective, since the vector spaces have the same dimension. This can be done by showing the kernel is trivial, which is done on the attached page.

Since $\mathbb{Q}$ is a splitting field for this group, we can compute the central idempotents using the formula on page 135 of Lam. As an example, the identity representation is

$$\frac{1}{6}\sum_{g\in G} g.$$

b. It suffices to find 4 degree 1 representations. Obviously, we have the trivial representation. Writing the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, we can take the alternating representation on either of the Z_2s and the trivial representation on the other, or that the alternating representation on both Z_2s. These are all 1-dimensional, so this gives the decomposition $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$ of the group algebra.

Since $\mathbb{Q}$ is a splitting field for this group, we can compute the central idempotents using the formula on page 135 of Lam. As an example, the trivial representation is

$$\frac{1}{4}\sum_{g\in G} g.$$

c. We obviously have the trivial representation.

If N is a normal subgroup of G and we have a representation ϕ on G/N, we can lift it to G by setting $\bar{\phi}(g) = \phi(gN)$. Note that the lifted representation is irreducible if the original one is. There are three subgroups $\{1, i, -1, -i\}$, $\{1, j, -1, -j\}$, and $\{1, k, -1, -k\}$, which gives 1 dimensional representations on the entire group lifted from the the alternating representation on $\mathbb{Z}_2$, which all must be irreducible.

We also a faithful representation by sending elements to the corresponding element in the 4-dimensional quaternion algebra over Q. This gives the putative decomposition $\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}\times Q$. One can show this is actually correct by working as in part (a). I omit the tedious calculation. (That the faithful representation is irreducible is then a consequence of Wedderburn's theorem.)

I don't see a good way to compute the central idempotents here, because $\mathbb{Q}$ is not a splitting field for this group.

7. We have the trivial representation of dimension 1, the standard representation of dimension 4, and the permutation representation of dimension 5. One can tediously but easily compute the characters and see that they have norm 1, so the corresponding representations are irreducible over $\mathbb{C}$ and hence over $\mathbb{Q}$. The interesting part of this problem is that A_5 has two irreducible characters of dimension 3 over $\mathbb{C}$ but a single irreducible representation of dimension 6 over $\mathbb{Q}$. We need to prove this.

   Work first over $\mathbb{C}$. Using the fact that the sum of the squares of the dimensions of the irreducible representations sum to the order of the group (a fact which is only true over an algebraically closed field), we get $1 + 16 + 25 + x^2 + y^2 = 60$, where we know there are five irreducible characters because A_5 has 5 conjugacy classes. Then $x^2 + y^2 = 18$ and the only possibility is $x = y = 3$.

   We can realize A_5 as the group of rotations of the isosahedron. In particular, one 3-dimensional irreducible representation corresponds to sending (12)(34) to the rotation by 180 degrees around an axis perpendicular to two opposite edges, (123) to the rotation by 120 degrees around an axis through two opposite pages, and (12345) and (13254) to rotations by 72 degrees and 144 degrees around axes through opposite verticies. This information comes from Etingof's Introduction to Representation Theory. We can view this as acting on a basis of $\mathbb{R}^3$ (see next problem) and use the fact that a rotation through angle θ has trace $1 + 2\cos\theta$. The trigonometric functions of angles that are multiples of 72 degrees are easily computable using a 36-72-72 triangle by drawing an angle bisector through one of the larger angles and are known to lie in $\mathbb{Q}(\sqrt{5})$. The character for the other 3-dimensional representation can be found by applying the conjugation $\sqrt{5} \to -\sqrt{5}$ to this character, and is an irreducible character because it still has norm 1.

Note that the field extension that the irreducible representations lie is in $\mathbb{Q}(\sqrt{5})$, since any larger extension would admit more conjugates of the degree 3 irreducible representations, and we know there are only two.

We want to show that the direct sum of the two irreducible dimension three representations over $\mathbb{C}$ is an irreducible representation over $\mathbb{Q}$. The matrices of this direct sum look like two diagonal blocks, the second the Galois conjugate of the first. The matrices are six dimensional, so let basis vectors v_1, v_2, v_3 correspond to the first block and v_4, v_5, v_6 to the second. Then if we change basis so $e_1 = v_1 + v_4$ and $e_4 = \sqrt{5}(v_1 - v_4)$ (and so on), we see we get rational matrices, so the representation is realizable over the rationals.

It must be irreducible because if it weren't, it would have to split into smaller representations over $\mathbb{Q}$, and these are automatically representations over $\mathbb{C}$. But we know that it splits into exactly two representations over $\mathbb{C}$, ands that these aren't realizable over $\mathbb{Q}$ (because of their irrational traces), so the representation is irreducible.

8. it suffices to compute the character of the representation of the group of rotational symmetries of the cube and show it matches the given one, since two representations are equivalent if and only if they have the same character.

   Let the center of the cube be placed at the origin and let the basis vectors of v_1, v_2, v_3 of $\mathbb{R}^3$ run through the centers of three sides adjacent to a single corner. Letting S_4 act on the long diagonals of the cube gives a permutation representation on these three vectors. To compute the character of this representation, it suffces to compute the trace of one element of each conjugacy class. On the identity we clearly have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   with trace 3. Elements of order two correspond to 180-degree rotations through lines joining centers of two opposite edges. Taking the one through the edge between the face corresponding to v_1 and v_2 gives the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   with trace -1. Similarly, all matrices correponding to the conjugacy classes $(\cdot\cdot)(\cdot\cdot)$ and $(\cdot\cdot)$ have trace -1. Elements of order 3 correspond to rotations through long diagonals. Choosing one through the corner adjacent to all v_i gives a permutation of the v_i with no fixed points, so the corresponding matrix has trace 0. Finally, a 4-cycle corresponds to a rotation through a line between the center of two opposite faces. Pick the line containing v_1. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

   with trace 1. This gives the character of the given representation, so we are done.

9. Let ω be a third root of unity. Conjugating one of the given representations and adding the trivial representation gives two more characters. Note that since a linear character can be viewed as a homomorphism to $\mathbb{C}^*$, multiplying a linear character with an irreducible character yields another irreducible character. One can use the orthoganality relation to check this, noting that any linear

character takes values with absolute value 1, so its conjugate is its inverse. This gives two more characters. Then the orthogonality relations suffice to fill in the final row.

1	1	1	1	1	1	1
1	1	1	ω^2	ω	ω^2	ω
1	1	1	ω	ω^2	ω	ω^2
2	-2	0	-1	-1	1	1
2	-2	0	$-\omega$	$-\omega^2$	ω	ω^2
2	-2	0	$-\omega^2$	$-\omega$	ω^2	ω
3	3	-1	0	0	0	0