Problem Set #11 Solutions

1\. Solve: Suppose $n = p_1^{e_1} \cdots p_m^{e_m}$.

If $\bar{k} \in \mathrm{Rad}(0)$, which means $\exists\, l$ s.t. $\bar{k}^l = \bar{0}$ in $\mathbb{Z}_n$,

$k^l \in n\mathbb{Z} \Rightarrow p_i \mid k^l \ \forall\, 1 \le i \le m \Rightarrow p_i \mid k \ \forall\, 1 \le i \le m \Rightarrow \bar{k} \in (p_1 \cdots p_m)$.

If $\bar{k} \in (p_1 \cdots p_m)$, $\bar{k}^{\max(e_1, \dots, e_m)} \in (p_1^{e_1} \cdots p_m^{e_m}) = (0)$.

In conclusion, $\mathrm{Rad}(0) = (p_1 \cdots p_m)$.

2\. Proof: Suppose I is a primary proper ideal. We know that I is included in a maximal ideal, which is of the form (c) for some c with $c^2 = c$.

Since (c) is proper, $1 \notin (c)$ thus $1 - c \notin (c)$, $1 - c \notin I$.

$0 = c(1-c) \in I$, $1 - c \notin I \Rightarrow \exists\, n$ s.t. $c^n \in I$, according to I is primary.

By $c^2 = c$, we have $c^n = c$, $c \in I$, thus $I = (c)$.

This means every primary ideal is finitely generated, so does every prime ideal.

By Cohen's theorem, R is Noetherian. $\square$

3\. Proof:

For $f \in \mathrm{Rad}((x^i, y^j))$, $f^n = gx^i + hy^j \Rightarrow f^n(0,0) = 0 \Rightarrow f(0,0) = 0 \Rightarrow f \in (x, y)$.

For $f \in (x, y)$, $f = gx + hy$, $f^{i+j} = \sum_{k=0}^{i} \left(\binom{i+j}{k} (gx)^k (h)^{i+j-k} y^{i-k}\right) y^j + \sum_{k=i+1}^{i+j} \left(\binom{i+j}{k} g^k x^{k-i} (hy)^{i+j-k}\right) x^i \in (x^i, y^j)$

Thus $\mathrm{Rad}((x^i, y^j)) = (x, y)$ $\square$

4\. Solve: $I = (x, z) \cap (x^2, y, z)$ OK

$\mathrm{Rad}((x, z)) = (x, z)$ $\quad \mathrm{Rad}((x^2, y, z)) = (x, y, z)$.

5\. Proof: (a) For $I \subset P$. Let $\mathcal{P} := \{P' \mid P' \text{ prime}, I \subset P' \subset P\}$. $(\mathcal{P}, \supset)$ is a partial order set.

and let $\{P_n\} \subset \mathcal{P}$ s.t. $P_1 \supset P_2 \supset \cdots$

Consider $\bigcap_{n=1}^{\infty} P_n$. Easy to see it is an ideal and $I \subset \bigcap_{n=1}^{\infty} P_n \subset P$.

Suppose $r_1 r_2 \in \bigcap_{n=1}^{\infty} P_n$ and $r_1 \notin \bigcap_{n=1}^{\infty} P_n$. This means $\forall\, i$, $r_1 r_2 \in P_i$ and $\exists\, j$ s.t. $r_1 \notin P_j$

By P_j prime, $r_2 \in P_j$ thus for $k \le j$, $r_2 \in P_k$.

For $k > j$, since $r_1 \notin P_j$, $r_1 \notin P_k$, thus by P_k prime, $r_2 \in P_k$.

In conclusion, $r_2 \in \bigcap_{n=1}^{\infty} P_n$, which means $\bigcap_{n=1}^{\infty} P_n$ is prime,

$\bigcap_{n=1}^{\infty} P_n \in \mathcal{P}$. By Zorn's lemma, $\mathcal{P}$ has a maximal element.

This means there is a minimal prime ideal of I. $\square$

(b) Every proper ideal is contained in a maximal ideal and it is prime.

By (a), it has at least one minimal prime ideal. $\square$

(c) By $\mathrm{Rad}\, I = \bigcap_{\substack{P \supset I \\ P \text{ prime}}} P$, and for each $I \subset P$ we have a minimal prime ideal of I.

This means we can replace each P with all the minimal prime ideals of I included in P.

Thus $\mathrm{Rad}\, I = \bigcap_{P \supset I} P$. $\square$

6. Proof. Suppose $rx \in N$ and $x \notin N$. By N is P-primary, we have

$\exists n$ s.t. $r^n M \subset N$, $r^n \in \mathrm{ann}(M/N)$, $r \in \mathrm{Rad}(\mathrm{ann}(M/N)) = P$.

Thus we have either $r \in P$ or $x \in N$. $\square$

7. Proof. (a). $\forall r_1 \in \mathrm{ann}(x)$, $r_2 \in R$, $(r_2 r_1)x = r_2(r_1 x) = r_2 0 = 0$, $r_2 r_1 \in \mathrm{ann}(x)$.

Thus $\mathrm{ann}(x)$ is an ideal.

(b). Let P be a maximal element of $\{\mathrm{ann}(x) \mid x \in M \setminus \{0\}\}$,

and say $P = \mathrm{ann}(x_0)$.

Suppose $r_1 r_2 \in P$ and $r_1 \notin P$.

By $r_1 r_2 x_0 = 0$, ~~[illegible]~~ $r_1 \in \mathrm{ann}(r_2 x_0)$.

According to $\mathrm{ann}(r_2 x_0) \supset \mathrm{ann}(x_0)$ and $r_1 \in \mathrm{ann}(r_2 x_0) \setminus \mathrm{ann}(x_0)$,

$\mathrm{ann}(r_2 x_0) \supsetneq \mathrm{ann}(x_0)$. By the maximum of P,

We must have $r_2 x_0 = 0$, $r_2 \in \mathrm{ann}(x_0) = P$.

Thus P is prime. $\square$

8. Proof. (a) $\{\{\mathrm{ann}(x)\}, \subset\}$ is a partial order set and according to

R is Noetherian, each $\{I_n\} \subset \{\mathrm{ann}(x)\}$ s.t. $I_1 \subset I_2 \subset \cdots$

must have ~~an~~ $n \in \mathbb{N}$ s.t. $\forall i > n$, $I_i = I_n$.

By Zorn's lemma, $\{\mathrm{ann}(x)\}$ has a maximal element and

by 7(b) it is prime, thus an associated prime exists.

(b) Choose $\tilde{P}_1$ be an associated prime of M.

Let $\tilde{P}_1 = \mathrm{ann}(x_1)$.

We have $\tilde{M}_1 = Rx_1 \cong R/\tilde{P}_1$.

If $M/\tilde{M}_1 \neq \{0\}$, Choose $\tilde{P}_2$ be an associated prime for $M/\tilde{M}_1$.

Let $\tilde{P}_2 = \mathrm{ann}(\bar{x}_2)$. Choose x_2 be a preimage of $\bar{x}_2$ and we have $\tilde{P}_2 = \mathrm{ann}(x_2)$ in M.

$\tilde{M}_2 := Mx_1 + Mx_2$, $\tilde{M}_2/\tilde{M}_1 = Rx_1 + Rx_2/Rx_1 = Rx_2/Rx_1 \cap Rx_2 = Rx_2 \cong R/\tilde{P}_2$.

If $M/\tilde{M}_r$ never reaches $\{0\}$, By induction we can construct a sequence

$\{0\} \subset \tilde{M}_1 \subset \tilde{M}_2 \subset \cdots$ and by M satisfies ACC, $\exists r$ s.t. $\forall i > r$, $\tilde{M}_i = \tilde{M}_r$.

Since $\tilde{M}_{r+1}/\tilde{M}_r = R/\tilde{P}_{r+1}$, $\tilde{P}_{r+1} = R$, $\mathrm{ann}(x_{r+1}) = R$, which is a contradiction.

~~[illegible]~~

Thus we must have some $r \in \mathbb{N}$ s.t. $M/\tilde{M}_r = \{0\}$.

$M = \tilde{M}_r \supset \tilde{M}_{r-1} \supset \cdots \supset \tilde{M}_1 \supset \{0\}$ and $\tilde{M}_i/\tilde{M}_{i-1} \cong R/\tilde{P}_i$.

9. Proof. (i) $\Rightarrow$ (ii). Let P be an associated prime of M. $P = \mathrm{ann}(x)$.

$\exists\, n(x) \in \mathbb{N}$ s.t. $r^{n(x)} x = 0$. $r^{n(x)} \in \mathrm{ann}(x) = P$. By P prime. $r \in P$.

Thus r lies in every associated prime of M.

(ii) $\Rightarrow$ (i). $\forall x \in M$. write $P = \mathrm{Rad}(\mathrm{ann}(x))$.

Since P is prime and R Noetherian. P is finite generated.

Let $P = (a_1, \cdots, a_n)$.

Since $P = \mathrm{Rad}(\mathrm{ann}(x))$. we have $n_i \in \mathbb{N}$ s.t. $a_i^{n_i} x = 0$.

Consider $x_0 = \left(\prod_{i=1}^{n} a_i^{n_i - 1}\right) \cdot x$.

We have $P = \mathrm{ann}(x_0)$. Which means

P is an associated prime of M. $r \in P$.

$r \in \mathrm{Rad}(\mathrm{ann}(x))$. $\exists\, n(x)$ s.t. $r^{n(x)} x = 0$. $\square$.