Problem Set 12 (Sel-hung)

① a. Prove that $\mathbb{C}[z]$ is integrally closed (in its field of fractions $\mathbb{C}(z)$).

Let $\frac{P}{Q} \in \mathbb{C}(z)$, where $P, Q \in \mathbb{C}[z]$, $(P,Q)=1$, and suppose $F(P/Q)=0$ for some $F=\sum_{i=1}^{n} h_i X^i \in \mathbb{C}[z][X]$. Then

$$h_n\left(\frac{P}{Q}\right)^n + \ldots + h_1\frac{P}{Q} + h_0 = 0$$

$$\Rightarrow \quad h_n P^n = Q\left(\sum_{i=0}^{n-1}(-h_i)P^i Q^{n-i-1}\right).$$

Since $Q \nmid P^n$, $Q \mid h_n$ and

$$h_n = 1 \text{ (F is monic)} \Longrightarrow Q = 1 \Rightarrow \frac{P}{Q} \in \mathbb{C}[z].$$

Hence every element in $\mathbb{C}(z)$ integral over $\mathbb{C}[z]$ is in $\mathbb{C}[z]$.

b. Prove the same result for any UFD.

The argument in a. holds mutatis mutandis for any corresponding h_i, P, and Q in a UFD with its field of fractions.

2. a. Note that

$$\frac{\mathbb{C}[x,y,z]}{(y^2 - x^3 + x^2, z^2 - x + 1, z^3 + z - y)} \cong \frac{\mathbb{C}[x,z]}{((z^3+z)^2 - x^3 + x^2, z^2 - x + 1)}$$

since the relation $z^3 + z - y$ allows us to remove all instances of y. Similarly, the relation $x = z^2 + 1$ allows us to remove all instances of x. Since $(z^3 + z)^2 - x^3 + x^2 = 0$ under this relation, we get the ring $\mathbb{C}[z]$.

b. View S as $\mathbb{C}[z]$. We have $\phi(x) = z^2 + 1$ and $\phi(y) = z^3 + z$. We need to show that no polynomial $p(x,y) \in \mathbb{C}[x,y]/(y^2 - x^3 + x^2)$ is sent to 0 by ϕ. Such a polynomial may be written as $yr(x) + s(x) + c$. After applying ϕ, this is $z(z^2+1)r(z^2+1) + s(z^2+1) + c$. Putting in $z = i$, we see that the constant term of s is $-c$. We can rewrite this as $z(z^2+1)r(z^2+1) + (z^2+1)t(z^2+1)$, where t has no constant term. The first term has odd degree and the second even degree, so they cannot cancel unless each is zero.

<u>Note</u>: The geometric map is $z \longmapsto (z^2+1, z^3+z) =: (x(z), y(z))$

$$\mathbb{C} \twoheadrightarrow E := \{(x,y) \in \mathbb{C}^2 \mid y^2 = x^3 - x^2\}$$

c. As before, we see that S' is isomorphic to

$$\frac{\mathbb{C}[y,z]}{(y^2 - (z^2+1)^3 + (z^2+1)^2)} = \frac{\mathbb{C}[y,z]}{(y^2 - (z^3+z)^2)}.$$

By the Chinese Remainder theorem, this is

$$\frac{\mathbb{C}[y,z]}{(y + z^3 + z)} \times \frac{\mathbb{C}[y,z]}{(y - z^3 - z)}$$

since the ideals are comaximal. This is manifestly not a domain: $(z, 0) \cdot (0, z) = 0$.

d. Clearly S lies inside of R's field of fractions. It is integrally closed ($\cong \mathbb{C}[z]$) and all of its elements are elements of the integral closure of R, so it is the integral closure of R.

3. Let N be the ideal of R generated by the a_i. The hypothesis implies

$$\mathfrak{m} = N + \mathfrak{m}\mathfrak{m}.$$

Nakayama's lemma then implies that $N = \mathfrak{m}$, as desired.

④ Let S be an integral extension ring of R and suppose R and S are domains. Show that S is a field $\Leftrightarrow$ R is a field.

($\Rightarrow$) Suppose S is a field, and let $r \in R$.
Then $\exists\, r^{-1} \in S$ (since $r \in S$) and r^{-1} is integral over R,
i.e.

$$(r^{-1})^n + r_{n-1}(r^{-1})^{n-1} + \ldots + r_1 r^{-1} + r_0 = 0, \quad r_i \in R.$$

$$\underset{\text{by } r^{n-1}}{\overset{\text{mult}}{\Longrightarrow}} r^{-1} = r_{n-1} + r_{n-2} r + \ldots + r_1 r^{n-2} + r_0 r^{n-1} \in R,$$

so $r^{-1} \in R$.

($\Leftarrow$) Suppose R is a field, and let $s \in S$.
Then by integrality

$$s^n + r_{n-1} s^{n-1} + \ldots + r_1 s + r_0 = 0$$

$$\Rightarrow s\left[(-r_0^{-1})(s^{n-1} + r_{n-1} s^{n-2} + \ldots + r_1)\right] = 1,$$

so s has an inverse in S.

⑤ Show that every affine k-variety in K^n is of the form $V(S)$ where S is a finite subset of $k[X_1, \ldots X_n]$.

In class we showed that every affine k-variety corresponded to a radical ideal of $k[X_1, \ldots, X_n]$. By the Hilbert basis theorem $k[X_1, \ldots X_n]$ is Noetherian, and this in turn implies that all ideals are finitely generated. If S is the set of generators, the variety then gives $V(S)$.

6. Each V_i corresponds to a radical ideal I_i. Because the correspondence is inclusion-reversing, this gives an increasing sequence of ideal $\{i_i\}$. Such a sequence must stabilize, since $k[x_1, \ldots x_n]$ is Noetherian by the Hilbert basis theorem, so the corresponding Varieties must stabilize too.

7. a. By induction, it suffices to consider the case of two ideals, I and J. Since $I \cap J \subset I$ and $I \cap J \subset J$, $V(J) \subset V(I \cap J)$ and $V(I) \subset V(I \cap J)$, so $V(I) \cup V(J) \subset V(I \cap J)$. For the other direction, suppose $p \in V(I \cap J)$. Note that $IJ \subset I \cap J$, so p vanishes on every polynomial in IJ, so $p \in V(I) \cup V(J)$. If not, then there is some $f \in I$ and $g \in J$ such that p is not a root of either, so p is not a root of $fg \in IJ$, a contradiction.

b. By induction, it suffices to consider the case of two ideals, I and J. Suppose p is a point in $V(I+J)$. The ideal I has a finite generating set f_i and J a generating set g_i, and $V(I+J) = V(f_i, g_i)$. Again by induction, it suffices to show that $V(f,g) = V(f) \cap V(g)$. But this is obvious from the definition.

8. Proof. (a) ($\Rightarrow$) Let V be irreducible. $f_1 f_2 \in J(V)$. $f_1 \notin J(V)$. $f_2 \notin J(V)$.

This means we can write $J(V) = $ ~~[illegible]~~ $(f_1, J(V)) \cap (f_2, J(V))$

with $(f_1, J(V)) \neq J(V)$. $(f_2, J(V)) \neq J(V)$.

$V = V(J(V)) = V((f_1, J(V)) \cap (f_2, J(V))) = V((f_1, J(V)) \cup V((f_2, J(V))$

and $V \neq V((f_1, J(V))$, $V \neq V((f_2, J(V))$

contradict to V irred.

($\Leftarrow$) Let $J(V)$ be prime. $V = W_1 \cup W_2$

$J(V) = J(W_1 \cup W_2) = J(W_1) J(W_2)$

Since $J(V)$ prime, we have $J(W_1) = J(V)$ or $J(W_2) = J(V)$

which means $W_1 = V$ or $W_2 = V$. $\square$.

(b) Consider $x_1^2 - 2x_2^2 \in \mathbb{Q}[x_1, x_2]$. It is irreducible in $\mathbb{Q}[x_1, x_2]$

thus $(x_1^2 - 2x_2^2)$ is prime.

$V(x_1^2 - 2x_2^2)$ is irreducible as a $\mathbb{Q}$-variety.

In $\mathbb{R}[x_1, x_2]$, $x_1^2 - 2x_2^2 = (x_1 - \sqrt{2}x_2)(x_1 + \sqrt{2}x_2)$

is not irreducible which means $(x_1^2 - 2x_2^2)$ is not prime

thus $V(x_1^2 - 2x_2^2)$ is not irreducible as a $\mathbb{R}$-variety.