HW #5 Solution

1. From the derivative test and the assumption on the characteristic of the field, $x^p - a$ has distinct roots. Consider two such roots r and s and consider r/s. This is in the field and $(r/s)^p = 1$, so F contains a pth root of unity not equal to 1. By taking powers, we get all pth roots of unity as p is prime. By lemma 2 on page 253, the Galois group of $x^p - a$ over F is $\mathbb{Z}_p$ or trivial. The first case corresponds to $x^p - a$ being irreducible. Since we already have a pth root of unity, to get the splitting field of $x^p - a$ we just need to adjoint a pth root of a. That the degree of $F(\sqrt[p]{a})/F$ is p shows that the minimal polynomial for this root has degree p and must be $x^p - a$, so it is irreducible. In the second case, F contains a root of $x^p - a$.

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2. The splitting field is $\mathbb{Q}(\omega, \sqrt[p]{p})$, where the pth root is taken to be real and ω is a primitive pth root of unity. Considering topmost extension of tower of extensions $\mathbb{Q}(\omega, \sqrt[p]{p})/\mathbb{Q}(\omega)/\mathbb{Q}$ shows that the Galois group of the polynomial has a subgroup $N \cong \mathbb{Z}_p$ that sends $\omega^y \sqrt[p]{a}$ to $\omega^{y+c} \sqrt[p]{a}$ and fixes the cyclotomic field by lemma 2 on page 253. Considering $\mathbb{Q}(\omega, \sqrt[p]{p})/\mathbb{Q}(\sqrt[p]{p}))/\mathbb{Q}$ shows there is a subgroup $M \cong \mathbb{Z}_{p-1}$ of the Galois group that acts by exponentiation on ω. Consider the group NM and consider its action on $\omega^y \sqrt[p]{a}$. This is sent to $\omega^{by+c} \sqrt[p]{a}$ for $b \in \mathbb{Z}_p \setminus \{0\}$ and $c \in \mathbb{Z}_p$, and clearly we can get all such transformations. This is the entire Galois group, since it has order $p(p-1)$, so the Galois gropu is isomorphic to the desired group affine transformations.

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3. The discriminant vanishes if and only if the equation has root with multiplicity greater than 1. This is clear from the definition. If all roots are real, then the discriminant is the product of squares of real numbers and must be real and positive. If the equation has one real root and two complex roots, we know the complex roots come in conjugate pairs. Call the real root r and the complex roots $a + ib$ and $a - ib$ for $a, b \in \mathbb{R}$. The discriminant is

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$$(a - r + ib)^2 (a - r - ib)^2 (2ib)^2.$$

The first two factors are the square of the norm of $a - r + ib$ and hence real and positive. The last factor is the square of an imaginary number and hence negative. So the entire product is negative.

4) Suppose $G \leq S_4$. G acts transitively on $\{1,2,3,4\}$.

By orbit-stablizer thm. $|G(x)|\,|G_x| = |G|$ then $4 \mid |G|$. so $|G| = 4, 8, 12, 24$

Case I. $|G| = 24$. $G \cong S_4$ of case ~~trivial~~ transitive.

σ	$\lvert ccl(\sigma) \rvert$
id	1
$(\cdot\cdot)$	6
$(\cdot\cdot\cdot)$	8
$(\cdot\cdot\cdot\cdot)$	6
$(\cdot\cdot)(\cdot\cdot)$	3

Case II: $|G| = 12$. In this case $G \trianglelefteq S_4$. $G = \coprod ccl(\sigma)$.

Easy to check the only possible choice of G is A_4.

A_4 is transitive.

Case III. $|G| = 8$. G is a Sylow 2-gp of S_4. So G must be

$D = \{1, \cancel{(12)(34)}\ (1234), (13)(24), (1432), (12), (34), (1423), (1324)\}$ or its conjugates. D is transitive, so is its conjugate.

Case IV: $|G| = 4$. $G \cong \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

For the Case $G \cong \mathbb{Z}_4$. The element with order 4 is S_4 are $(abcd)$

Thus $G = \langle (abcd) \rangle$ (conjugates with $C = \langle (1234) \rangle$).

and G is transitive.

For the case $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ elements of $G \setminus \{1\}$ must be $(\cdot\cdot)$ or $(\cdot\cdot)(\cdot\cdot)$

For the case $G = \{1, (12)(34), (13)(24), (14)(23)\}$. G is Transitive

For the case $G = \{1, (\cdot\cdot), (\cdot\cdot)(\cdot\cdot), (\cdot\cdot)(\cdot\cdot)\}$. G cannot be a group.

For the case $G = \{1, (ab), (cd), (ab)(cd)\}$. G is NOT transitive

To conclude. the possibles ones are $S_4, A_4, D_4, \mathbb{Z}_4, V_4$

5. $p=-2,\ q=4\quad \Delta=-4p^3-27q^2=-400.$

$z_1=\sqrt[3]{-\frac{27}{2}q+\frac{3}{2}\sqrt{-3\Delta}}=\sqrt[3]{-54+30\sqrt{3}}\ ,\ z_2=\sqrt[3]{-\frac{27}{2}q-\frac{3}{2}\sqrt{-3\Delta}}=\sqrt[3]{-54-30\sqrt{3}}$

$(-3+\sqrt{3})^3=-27+3\cdot(-3)\sqrt{3}^2+3\cdot(-3)^2\cdot\sqrt{3}+3\sqrt{3}=-54+30\sqrt{3}$

$(-3-\sqrt{3})^3=-27+3(-3)\sqrt{3}^2-3\cdot(-3)^2\sqrt{3}-3\sqrt{3}=-54-30\sqrt{3}.$

$\frac{1}{3}(z_1+z_2)=\frac{1}{3}(-3+\sqrt{3}-3-\sqrt{3})=-2.$

$\frac{1}{3}(\zeta_3^2z_1+\zeta_3z_2)=\frac{1}{3}(-3(\zeta_3^2+\zeta_3)+\sqrt{3}(\zeta_3^2-\zeta_3))=\frac{1}{3}(3-3i)=1-i.$

$\frac{1}{3}(\zeta_3z_1+\zeta_3^2z_2)=\frac{1}{3}(-3(\zeta_3+\zeta_3^2)+\sqrt{3}(\zeta_3-\zeta_3^2))=\frac{1}{3}(3+3i)=1+i.$

The roots of $X^3-2X+10=0$ are $2,\ 1-i,\ 1+i.$

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$H\cong D_5$ and both Z_5 and $D_5/Z_5\cong Z_2$ are solvable then H is solvable.

$H=\langle(12345),(14)(23)\rangle$. So we need to check w_1 is fixed under (12345) and $(14)(23)$. $(12345)w_1=X_2X_3+X_3X_4+X_4X_5+X_5X_1+X_1X_2-X_2X_4-X_3X_5-X_4X_1-X_5X_2-X_1X_3=w_1$ & $(14)(23)w_1=X_4X_3+X_3X_2+X_2X_1+X_1X_5+X_5X_4-X_4X_2-X_3X_1-X_2X_5-X_1X_4-X_5X_3=w_1$. So w_1 is fixed under H

The conjugate of w_1 under $\mathrm{Aut}(F(X_1,\dots,X_5)/K)$ are $w_2=X_1X_3+X_3X_5+X_5X_2+X_2X_4+X_4X_1-X_1X_5-X_3X_2-X_5X_4-X_2X_1-X_4X_3$ $w_3=X_2X_4+X_4X_5+X_5X_3+X_3X_1+X_1X_2-X_2X_5-X_4X_3-X_5X_1-X_3X_2-X_1X_4$ $w_4=X_2X_1+X_1X_4+X_4X_3+X_3X_5+X_5X_2-X_2X_4-X_1X_3-X_4X_5-X_3X_2-X_5X_1$ $w_5=X_1X_5+X_5X_4+X_4X_2+X_2X_3+X_3X_1-X_1X_4-X_5X_2-X_4X_3-X_2X_1-X_3X_5.$

$w_6=X_2X_5+X_5X_1+X_1X_3+X_3X_4+X_4X_2-X_2X_1-X_5X_3-X_1X_4-X_3X_2-X_4X_5.$

By orbit-stablizer thm, the stablizer of w_1 is $\mathrm{Aut}(F(X_1\dots X_5)/K)$ has order 10, which is H. $[K(w_1):K]=\dfrac{[F(X_1,\dots,X_5):K]}{[F(X_1,\dots,X_5):K(w_1)]}=\dfrac{|A_5|}{|H|}=\dfrac{60}{10}=6$

7. By Newton's identities we have $\lambda_1=\lambda_2=\lambda_3=0$. $\lambda_4=(-1)^4\cdot4(-p)=-4p$.

$\lambda_5=(-1)^5\cdot5\cdot q=-5q$. $\lambda_6=\lambda_7=0\quad\lambda_8=(-1)^4(-p)\cdot(-4p)=4p^2.$

$$\Delta=\begin{vmatrix}5&0&0&0&-4p\\0&0&0&-4p&-5q\\0&0&-4p&-5q&0\\0&-4p&-5q&0&0\\-4p&-5q&0&0&4p^2\end{vmatrix}=5\cdot\begin{vmatrix}0&0&-4p&-5q\\0&-4p&-5q&0\\-4p&-5q&0&0\\-5q&0&0&4p^2\end{vmatrix}+(-4p)\cdot\begin{vmatrix}0&0&0&-4p\\0&0&-4p&-5q\\0&-4p&-5q&0\\-4p&-5q&0&0\end{vmatrix}$$

$$-5(-(-5p^3))\cdot\begin{vmatrix} 0 & -4p & -5q \\ -4p & -5q & 0 \\ -5q & 0 & 0 \end{vmatrix} + 5\cdot 4p^2\cdot\begin{vmatrix} 0 & 0 & -4p \\ 0 & -4p & -5q \\ -4p & -5q & 0 \end{vmatrix} - 4\cdot 2^8 p^5$$

$$= 5^5 q^4 + (5-4)2^8 p^5 = 2^8 p^5 + 5^5 q^4.$$

Let $\mu_1, \dots, \mu_k$ be similar on the last question.

Then $\sum_{i=1}^{k} \mu_i^2 = 10p$, $\sum_{1\le i<j\le k} \mu_i^2\mu_j^2 = 55p^2$, $\sum \mu_i^2\mu_j^2\mu_s^2 = 140p^3$, $\sum \mu_i^2\mu_j^2\mu_s^2\mu_t^2 = 175p^4$.

$\sum \mu_i\mu_j\mu_s\mu_t\mu_r = 106p^5 + 5^5 q^4$. $\prod_{i=1}^{5} \mu_i = 5p^3$. so μ_i^2 are roots of

$(x^3 - 5px^2 + 5p^2x + 5p^3)^2 - dx = 0$.

By $\mathrm{Aut}(F(x_1, \dots, x_5)/F(\sqrt{d}, \mu_1)) \cong D_5$ is solvable. f is solvable by radicals$/F(\sqrt{d}, \mu_1)$.

8. First notice that $\eta_n = \begin{vmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{vmatrix}_{n\times n}$

$\lambda_k = \sum_{i=1}^{k} (-1)^{i+1} P_i \lambda_{k-i}$ for $1 \le k \le n$. $\lambda_{n+j} = \sum_{i=1}^{n} (-1)^{i+1} P_i \lambda_{n+j-i}$

For $k = 1, \dots, n-2$ we have $P_1 = \dots = P_{n-2} = 0$ they $\lambda_k = 0$.

$\lambda_{n-1} = (-1)^n P_{n-1}\lambda_0 = (-1)^n (n-1)(-1)^{n-1} p = -(n-1)p$.

$\lambda_n = (-1)^{n+1} n (-1)^n q = -nq$.

and $\lambda_{n+1} = \dots = \lambda_{2n-3} = 0$. since for $n+j \le 2n-1$, either $P_i = 0$ or $\lambda_{n+j-i} = 0$.

$\lambda_{2n-2} = (-1)^n P_{n-1}\lambda_{n-1} = (-1)^n \cdot (-1)^{n-1} p \cdot (-(n-1)p) = (n-1)p^2$.

$$\Delta = \begin{vmatrix} n & & & -(n-1)p \\ & 0 & \iddots & -nq \\ & \iddots & 0 & \\ -(n-1)p & -nq & \cdots & (n-1)p^2 \end{vmatrix} = n\cdot\begin{vmatrix} & & -(n-1)p & -nq \\ & \iddots & \iddots & \\ -(n-1)p & \iddots & & \\ -nq & & & (n-1)p^2 \end{vmatrix}_{(n-1)\times(n-1)} + (-1)^n (n-1)p \begin{vmatrix} & & & -(n-1)p \\ & & \iddots & -nq \\ & \iddots & & \vdots \\ -(n-1)p & -nq & & \end{vmatrix}$$

$$= -n^n q^{n-1}\eta_{n-1} + (n-1)^{n-1} p^n \eta_{n-2}$$

$$= \eta_{n+1} n^n q^{n-1} - \eta_n (n+1)^{n-1} p^n.$$

9. (a) $\Delta = 4^4(-12)^3 - 3^3\cdot 8^4 = -552960 \notin \mathbb{Q}^2$.

$f = x^3 + 48x - 64$ irred in $\mathbb{Q}[x] \Rightarrow \mathrm{Gal}_{\mathbb{Q}}(x^4 + 8x - 12) \cong S_4$

(b) $\Delta = 4^4 \in \mathbb{Q}^2$. $f = x^3 - 4y$ reducible in $\mathbb{Q}[x]$.

$\mathrm{Gal}_{\mathbb{Q}}(x^4 + 1) \cong V_4$.

(c) SFE of $x^4+x^3+x^2+x+1/\mathbb{Q}$ in $\mathbb{Q}(\zeta_5)$

$\mathrm{Aut}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong \mathbb{Z}_4$

$\mathrm{Gal}_{\mathbb{Q}}(x^4+x^3+x^2+x+1) \cong \mathbb{Z}_4$

(d) $\Delta = 4^4 \cdot (-2)^3 \notin \mathbb{Q}^2$ $\quad F = x^3+8x$ reducible in $\mathbb{Q}[x]$

Let M be SFE of F. $\quad M = \mathbb{Q}(\sqrt{2})$

f is irred over $M \Rightarrow \mathrm{Gal}_{\mathbb{Q}}(x^4-2) \cong D_4$