Problem Set 8 (Solutions)

(1) Show that if B is non-degenerate, then for every linear transformation T of V into V there exists a unique linear transformation T' of V into V such that $B(Tx,y) = B(x,T'y)$ for all $x,y \in V$. Determine the matrix of T' in terms of the matrices of T and B relative to a base of V. Show that the map $T \mapsto T'$ is an anti-automorphism in the ring of linear transformations and that $(T')' = T$ for T if B is either symmetric or skew.

Proof. B is non-degenerate, hence for some basis e of V, $[B]_e$ is invertible. Then define T' to be satisfy $[T']_e = [B]_e^{-1} \cdot {}^t[T]_e \cdot [B]_e$ which has the desired property. Then we notice such T' is unique. Suppose not, there is another linear transformation T'' also has the same property, namely, for any $x,y \in V$, $B(Tx,y) = B(x,T'y) = B(x,T''y)$. Choose $y_0 \in V$ such that $T'y_0 \neq T''y_0$. Then for any $x \in V$, $B(x, T'y_0 - T''y_0) = 0$, i.e. $\mathrm{span}\langle T'y_0 - T''y_0\rangle$ is the right kernel of B, a contradiction.

For two linear transformations S and T, $B(STx,y) = B(Tx,S'y) = B(x,T'S'y)$, in other words, $(ST)' = T'S'$.To show $T \mapsto T'$ is surjective, let $\widetilde{T}$ be the linear transform satisfies that $[\widetilde{T}]_e = [B]_e {}^t[T]_e [B]_e^{-1}$ and we have $\widetilde{T} \mapsto T$. Hence the map is an anti-automorphism.

For any $x,y \in V$ and let B be symmetric, $B(T'x,y) = B(y,T'x) = B(Ty,x) = B(x,Ty)$; if B is alternating, $B(T'x,y) = -B(y,T'x) = -B(Ty,x) = B(x,Ty)$. These prove $(T')' = T$ if T is symmetric or skew. □

(2) Show that

$$b = \begin{pmatrix} 0 & 2 & -1 & 3 \\ -2 & 0 & 4 & -2 \\ 1 & -4 & 0 & 1 \\ -3 & 2 & -1 & 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

are cogredient in $M_4(\mathbb{Q})$ and find a matrix p such that $pb\,{}^tp = s$.

Proof. Note $b = -{}^tb$ and $\det(b) \neq 0$, b is thus the matrix of a non-degenerate alternating form, and then cogredient to s.

Let

$$\begin{aligned} u_1 &= {}^t[0, -\frac{1}{2}, 0, 0], \\ v_1 &= {}^t[1,0,0,0], \\ u_2 &= {}^t[2, \frac{1}{2}, 1, 0], \\ v_2 &= {}^t[-\frac{1}{6}, -\frac{1}{4}, 0, \frac{1}{6}] \end{aligned}$$

and the ordered set $\{u_1, v_1, u_2, v_2\}$ is a symplectic basis of $\mathbb{Q}^4$. Then let

$$M = [u_1, v_1, u_2, v_2] = \begin{pmatrix} 0 & 1 & 2 & -\frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{6} \end{pmatrix}$$

and ${}^tMbM = s$. □

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N (3) Let $(u_i.v_i)$ be a symplectic base for V and let U and U' be the subspaces spanned by the u_i and the v_i respectively. Let K be the subset of $Sp_n(\mathbb{F})$ of η which stabilize U and U'. Show that a linear transformation $\eta \in K$ iff its matrix relative to the base

$$(u_1, \dots, u_r, v_1, \dots, v_r) \tag{1}$$

has the form

$$\begin{pmatrix} A & 0 \\ 0 & ({}^tA)^{-1} \end{pmatrix}, \quad A \in GL_r(\mathbb{F}). \tag{2}$$

Note that K is a subgroup of $Sp_n(\mathbb{F})$.

Proof. Let e denote the basis $\{u_1, \dots, u_r, v_1, \dots, v_r\}$ and if $[\eta]_e$ has such form, it is obvious η stabilize U and U' and that $[\eta]_e$ is non-singular. It is left to check η is symplectic. Let $x, y \in V$ and the matrix of the symplectic form under e is

$$[B]_e = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

Then

$$\begin{aligned} B(\eta x, \eta y) &= {}^tx \begin{pmatrix} {}^tA & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & ({}^tA)^{-1} \end{pmatrix} y \\ &= {}^tx \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix} y = B(x,y). \end{aligned}$$

Hence $\eta \in K$.

On the other hand, if $\eta \in K$ which stabilize U and U', it has the form

$$[\eta]_e = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}, \quad M_1, M_2 \in GL_r(\mathbb{F}).$$

Moreover, since η is symplectic, we have ${}^t[\eta]_e[B]_e[\eta]_e = [B]_e$, thus we get ${}^tM_1M_2 = I_r$, $M_2 = ({}^tM_1)^{-1}$. ☐

(4) Give an example of a symplectic transformation having no fixed points except for the origin.

Consider (V, B) to be a symplectic space of dimension 2 and $\{u, v\}$ as the symplectic basis of V. Put $\eta = \tau_{u,1}\tau_{v,1}$, $\eta(x) = x + B(x,u)u + B(x,v)v$. η is a composite of symplectic transvections and thus symplectic. We claim η has no fixed points except for the origin. Let $x \in V$ such that $x + B(x,u)u + B(x,v)v = x$, then $B(x,u)u + B(x,v)v = 0$, hence $B(x,u) = B(x,v) = 0$. B is non-degenerate and we have $x = 0$.

(5) Let (u_i, v_i) be a symplectic base arranged as in (1) and let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{ij} \in M_r(\mathbb{F}).$$

Show that A is the matrix of a symplectic transformation iff the A_{ij} satisfy

$$\begin{aligned} {}^tA_{11}A_{22} - {}^tA_{21}A_{12} = 1_r = {}^tA_{22}A_{11} - {}^tA_{12}A_{21}, \\ {}^tA_{11}A_{21} - {}^tA_{21}A_{11} = 0_r = {}^tA_{22}A_{12} - {}^tA_{12}A_{22}. \end{aligned}$$

Proof. Let e and B be as in Problem (3). The linear transformation A represents is symplectic iff

$${}^tA[B]_eA = [B]_e,$$

namely,

$$\begin{pmatrix} -{}^tA_{21} & {}^tA_{11} \\ -{}^tA_{22} & {}^tA_{12} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$$

and we are done. □

(6) Finish the proof of the extension-of-basis lemma from the beginning of Monday's class: if (V, B) is a symplectic vector space, then a linearly independent, isotropic subset $\{u_1, \ldots, u_k\} \subset V$ may be extended to a symplectic basis of V. [Notes: in the proof, V was called U'' and the span of the u_i was R. Also, a set $S \subset U$ is isotropic if $B(S, S) = \{0\}$.]

Proof. Let (V, B) be a symplectic space of dimension $2r$ over $\mathbb{F}$ with char$\mathbb{F} \neq 2$, U is a subset of V. Put $R := U \cap U^{\perp}$. Then the alternating form on the quotient space $\overline{U} := U/R$ satisfying $B(u+R, u'+R) = B(u, u')$ is well-defined since for any $r, r' \in R$, $B(u+r, u'+r') = B(u, u') + B(u, r') + B(r, u') + B(r, r') = B(u, u')$. It is also non-degenerate, for that R is the kernel of $B|_U$.

Let $\{\overline{u_1}, \overline{v_1}, \ldots, \overline{u_k}, \overline{v_k}\}$ be a symplectic basis of $\overline{U}$ and $\{u_1, v_1, \ldots, u_k, v_k\}$ be any lift of them, which are also linearly independent. Let $U' = \mathbb{F}\langle u_1, v_1, \ldots, u_k, v_k\rangle$, and it is clear $\{u_1, v_1, \ldots, u_k, v_k\}$ is a symplectic basis of U'. We now have $U = R \oplus U'$ and we put $\{w_1, \ldots, w_m\}$ to be a basis of R. Hence $U' \cap R = \{0\}$ gives $U' \cap (U')^{\perp} = \{0\}$ in V; together with B is non-degenerate on V, we know $V = U' \oplus (U')^{\perp}$ and that both $B|_{U'}$ and $B|_{(U')^{\perp}}$ are non-degenerate. Therefore it is left to show $\{w_1, \ldots, w_m\}$ could be extended to a symplectic basis of $(U')^{\perp}$.

Let f_j be a functional on $(U')^{\perp}$ such that $f_j(w_i) = \delta_{ij}$ and since B is non-degenerated on $(U')^{\perp}$, one can find $s_j \in (U')^{\perp}$ such that $f_j(z) = B(z, s_j)$ for all $z \in (U')^{\perp}$. Hence we have $\{w_1, s_1, \ldots, w_m, s_m\}$ such that $B(w_i, s_j) = \delta_{ij}$ and it is straightforward that $\{s_i\}$ should be linear independent. Now let $t_1 = s_1$ and by induction we can make $t_k = s_k - \sum_{i=1}^{k-1} B(s_k, t_i)w_i$. Then we may check that $B(t_i, w_j) = 0$ and $B(t_i, t_j) = \delta_{ij}$. Therefore $W := \mathbb{F}\langle w_1, t_1, \ldots, w_m, t_m\rangle$ is a orthogonal sum of hyperbolic planes and $B|_W$ is non-degenerate, thus $(U')^{\perp} = W \oplus W^{\perp}$. One may choose a symplectic basis of $W^{\perp}$ and in together we extend $\{w_i\}$ to a symplectic basis of $(U')^{\perp}$. □

(7) Find the diagonal matrix d cogredient in $M_3(\mathbb{Q})$ to

$$s = \begin{pmatrix} -2 & 3 & 5 \\ 3 & 1 & -1 \\ 5 & -1 & 4 \end{pmatrix}.$$

Also determine a matrix p such that $ps^tp = d$.

Solution. Set

$$\begin{aligned} v_1 &= {}^t[1,0,0], \\ v_2 &= {}^t[\frac{3}{2},1,0], \\ v_3 &= {}^t[\frac{8}{11},-\frac{13}{11},1]; \end{aligned}$$

then $\{v_1, v_2, v_3\}$ is an orthogonal basis. Set ${}^tp = [v_1, v_2, v_3]$ and we have

$$ps^tp = \begin{pmatrix} -2 & 0 & 0 \\ 0 & \frac{11}{2} & 0 \\ 0 & 0 & \frac{97}{11} \end{pmatrix}$$

8. We know that B is cogredient to some diagonal matrix where each diagonal entry is 1, -1, or 0. If v is a basis vector corresponding to a 0 or 1 entry, $B(v,v)$ is 0 or negative, contradicting the positive definite hypothesis. So there is a basis where B is represented by the identity matrix.

Fix some basis v_i and let s be the matrix for B in this basis. We want lower triangular p so that $psp^T = 1$. By the remarks on page 345, it suffices to find an orthonormal basis such that the first vector is of the form c_1v_1, the second is of the form $c_1v_1 + c_2v_2$, and in the general the jth basis vector has the coefficient of v_i equal to 0 for $i > j$. Note that it suffices to find an orthogonal basis and then normalize.

We proceed inductively by choosing v_1 as our first basis vector u_1. Assume we have found n orthogonal vectors u_i satisfying the necessary conditions. Set

$$u_{n+1} = v_{n+1} - \sum_1^n B(v_{n+1}, u_i)u_i.$$

Then u_{n+1} is orthogonal to the first n u_i and the induction is complete.

9. Let s be the matrix with respect to the first base and r the matrix with respect to the second. Then if p is the change of basis matrix, $psp^T = r$. But the matrix of a positive definitive form with respect to an orthonormal basis is the identity, as in the prior problem, so $pp^T = I$ as desired.

Let n be an invertible square matrix. Clearly nn^T is positive definite, so by the prior problem there is a triangular (invertible) matrix p with $pnn^Tp^T = I$. We then have $n = p^{-1}(n^Tp^T)^{-1} = (p^{-1})(pn)$. Since the inverse of a triangular matrix is triangular, and (pn) is orthogonal because $pnn^Tp^T = I$, we are done.

[redacted]

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10\. a. Let $B(x,y) = \frac{1}{2}(H(x,y) + H(y,x))$ (this corrects a typo in the book). This is clearly symmetric and bilinear, and for $a \in K$ we have

$$B(ax, ay) = \frac{1}{2}(H(ax, ay) + H(ay, ax)) = \frac{1}{2}N(a)(H(x,y) + H(y,x)) = N(a)B(x,y).$$

b. We must verify $H(x,y)$ is Hermitian. Linearity in each component is obvious. Since 1 and i is a basis of K and $1 \in F$, it suffices to verify linearity and conjugate-linearity on F-multiples of i.

$$H(aix, y) = B(aix, y) - b^{-1}iB(aix, iy) = b^{-1}B(ax, iy) + iB(ax, y) = aiH(x,y)$$

A similar argument works for conjugate linearity.

c. If $H(x,y) = B(x,y) - b^{-1}iB(x,iy)$, then $H \to B$ sends this to

$$\frac{1}{2}(B(x,y) + B(y,x) - b^{-1}i(B(x,iy) + B(y,ix)))$$

Since $N(i)B(ix,y) = B(bx, iy)$ implies $B(ix,y) = -b^{-1}B(bx,iy)$, we see $B(x,iy) + B(y,ix)$ vanishes and we recover B.

Similarly, if $B(x,y) = \frac{1}{2}(H(x,y) + H(y,x))$, then $B \to H$ sends this to

$$\frac{1}{2}(H(x,y)+H(y,x)-b^{-1}iH(x,iy)-b^{-1}iH(iy,x)) = \frac{1}{2}(H(x,y)+H(x,y)+H(x,y)-H(y,x) = H(x,y).$$

So the maps are inverses.