Problem Set #9 (Solutions)

1\. Solve: $\mathbb{Z}$-modules are abelian groups, and simple $\mathbb{Z}$-modules are simple abelian groups, say $\mathbb{Z}_p$ (p prime).

By semisimple modules are the direct sum of simple modules.

Semisimple $\mathbb{Z}$-modules are of form $\bigoplus_{\alpha\in\Lambda} \mathbb{Z}_{p_\alpha}$ (Λ is an arbitrary index set).

2\. Proof: In the case $R \cong M_n(D)$, let F be the center of D. We know that F is a field. Need to show $F\cdot I_n$ is the center of $M_n(D)$.

$\forall C \in F\cdot I_n$, $A \in M_n(D)$, Suppose $C = c\cdot I_n$, $c\in F$.

$C\cdot A = c\cdot A = A\cdot c = A\cdot C$ thus $C \in Z(M_n(D))$. $F\cdot I_n \subseteq Z(M_n(D))$.

$\forall B \in Z(M_n(D))$, Suppose $B = \{b_{ij}\}$. We have

$B(de_{ij}) = (de_{ij})\cdot B$ $\forall d\in D, 1\le i,j\le n$ ($d\cdot e_{ij} = \begin{pmatrix} & \vdots & \\ \cdots & d & \cdots \\ & \vdots & \end{pmatrix}$)

which means $\begin{cases} b_{ij} = 0 & \text{for } i\neq j \\ b_{ij} \in Z(D) = F & \text{for } i=j \end{cases}$

And by $B(\sum_{i=1}^{n} e_{i1}) = (\sum_{i=1}^{n} e_{i1})B$ we have all these b_{ii} are same.

Thus $B \in F\cdot I_n$.

In conclusion, $Z(M_n(D)) = F\cdot I_n$ is a field.

For R is a semisimple ring, by Artin-Wedderburn's thm we have

$R \cong M_{n_1}(D_1)\times\cdots\times M_{n_r}(D_r)$

$Z(R) \cong Z(M_{n_1}(D_1))\times\cdots\times Z(M_{n_r}(D_r))$

$\cong Z(D_1)\times\cdots\times Z(D_r)$

is a finite direct product of fields. □.

3\. Proof: According to R is semisimple, by Artin-Wedderburn's thm we have

$R \cong M_{n_1}(D_1)\times\cdots\times M_{n_r}(D_r)$ and there are exactly r non-$\cong$ simple modules over R, called V_i.

By M is a finitely generated R-module, we can write M as

$M \cong \bigoplus_{i=1}^{r} V_i^{\oplus m_i}$.

Thus we have $E = \mathrm{End}_R(M) \cong \prod_{i=1}^{r} \mathrm{End}_R(V_i^{\oplus m_i}) \cong \prod_{i=1}^{r} M_{m_i}(\mathrm{End}_R(V_i))$

$\cong \prod_{i=1}^{r} M_{m_i}(D_i)$

Thus E is semisimple. □.

4, omitted

Muxi Li

$\Longrightarrow$ take $\underbrace{\operatorname{diag}\{a_1, \dots, a_n\}}_{\text{or if they aren't distinct, a Jordan matrix}} \in M_n(k)$

5. Proof. $\Longleftarrow$ Let r be the elmt with $m_r = (x-a_1) \cdots (x-a_n)$.

We have a series

$0 \subsetneq R(r-a_1)\cdots(r-a_{n-1}) \subsetneq \cdots \subsetneq R(r-a_1) \subset R$

and we can construct a c.s. with length $\geq n$.

Suppose $R \cong M_m(D)$. $\dim_k D = d$. Then we have

$n^2 = m^2 \cdot d$ and $m^2 \geq n^2$, thus $d = 1$. $m = n$. $R \cong M_n(k)$. $\square$

6. Proof. Suppose V is a semisimple $k[H]$-module.

Consider V as a $k[G]$-module and $W \subset V$ a ~~sub~~ $k[G]$-submodule.

By V is a semisimple $k[H]$-module, $V = W \oplus W'$ as $k[H]$-module.

Let $f_W : V \twoheadrightarrow W$, $(w, w') \mapsto w$. $\quad f_W \in \operatorname{Hom}_{k[H]}(V, W)$ $\quad f_W|_W = id_W$.

Define $\pi(v) := \frac{1}{[G:H]} \sum_{Hg \text{ coset of } H} g^{-1} f_W(g.v)$. $\left(\pi(v) \text{ is well-defined. Suppose } g' = hg. \; g^{-1} f_W(g.v) = g'^{-1} h f_W(h^{-1} g' v) = g'^{-1} f_W(g' v) \right)$.

① $\pi \in \operatorname{Hom}_{k[G]}(V, W)$.

Let $\gamma \in G$. $\pi(\gamma . v) = \frac{1}{[G:H]} \sum_{Hg \text{ coset of } H} g^{-1} f_W(g.\gamma v)$

$\overset{g\gamma = \eta}{=} \frac{1}{[G:H]} \sum_{H\eta \text{ coset of } H} \gamma \eta^{-1} f_W(\eta . v)$

$= \gamma \pi(v)$.

② $\pi(w) = \frac{1}{[G:H]} \sum_{Hg \text{ coset of } H} g^{-1} f_W(g.w) = \frac{1}{[G:H]} \cdot [G:H] \cdot w = w$

Thus $V = W \oplus \operatorname{Ker} \pi$

V is semisimple. $\square$