I. Holomorphic functions

I.A. Complex numbers

§ Cartesian form.

I.A.1. DEFINITION. A **field** is a set $\mathbb{F}$ with

- 2 commutative, associative binary operations $(+, \cdot)$
- identity elements 0 for "+", 1 for "·"
- inverses: $-a$ for "+", a^{-1} (or $\frac{1}{a}$) for "·" if $a \neq 0$
- distributive law.

$\mathbb{F}$ is **algebraically closed** if for every polynomial equation

$$a_n x^n + \cdots + a_1 x + a_0 = 0$$

with coefficients a_i in the field, there exists a solution in the field.

I.A.2. EXAMPLES. $\mathbb{R}$ is a field, but is not algebraically closed:

$$x^2 + 1 = 0$$

has no solution in $\mathbb{R}$. We can define an algebraically closed[1] field containing $\mathbb{R}$ by

$$\mathbb{C} := \{a + b\mathbf{i} \mid a, b \in \mathbb{R}\} \quad \left(\cong \mathbb{R}^2 \text{ as a set}\right)$$

with

- addition: $(a_1 + b_1\mathbf{i}) + (a_2 + b_2\mathbf{i}) := (a_1 + a_2) + (b_1 + b_2)\mathbf{i}$
- additive inverse $-(a + b\mathbf{i}) := (-a) + (-b)\mathbf{i}$;
- multiplication:

$$(a_1 + b_1\mathbf{i}) \cdot (a_2 + b_2\mathbf{i}) := (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\mathbf{i}$$

Here we treat $\mathbf{i}$ as formal solution to $x^2 = -1$, i.e. $\sqrt{-1}$.

[1]We won't prove this now.

- What about the multiplicative inverse?

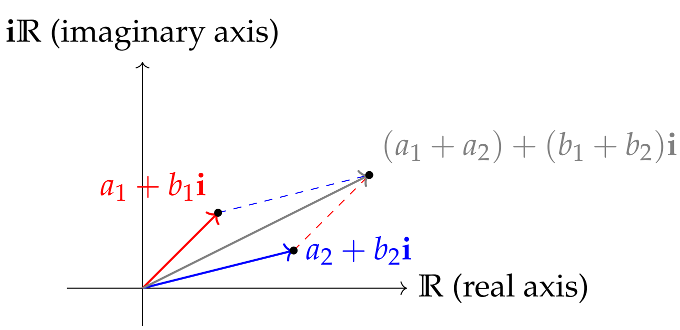
Notice that $(a+b\mathbf{i})(a-b\mathbf{i})=a^2+b^2$, and so

$$(a+b\mathbf{i})\left(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}\mathbf{i}\right)=1.$$

So the inverse $\frac{1}{a+b\mathbf{i}}$ (or $(a+b\mathbf{i})^{-1}$) is defined iff a,b both $\neq 0$. Finally,

- The "embedding map" $\mathbb{R}\hookrightarrow\mathbb{C}$ is given by $a\mapsto a+0\mathbf{i}$ and respects "$+$" and "$\cdot$" (i.e. is a "homomorphism of fields").

We visualize $\mathbb{R}\subset\mathbb{C}$ via the picture



in which addition corresponds to adding vectors in $\mathbb{R}^2$ — or, as we shall often call it, the **complex plane**.

I.A.3. DEFINITION. **Complex conjugation** flips the complex plane about the real axis: that is,

$$\alpha=a+b\mathbf{i}\quad\rightsquigarrow\quad\bar{\alpha}:=a-b\mathbf{i}.$$

The **real** and **imaginary parts** Re, Im are given by the projections to $\mathbb{R}$ and $\mathbf{i}\mathbb{R}$: in the above notation, $\mathrm{Re}(\alpha)=a$ and $\mathrm{Im}(\alpha)=b$.

Complex conjugation respects the operations "$+$" and "$\cdot$": we have $\overline{\alpha_1+\alpha_2}=\overline{\alpha_1}+\overline{\alpha_2}$ (think "add then flip" vs. "flip then add"), $\overline{\alpha_1\alpha_2}=\bar{\alpha}_1\cdot\bar{\alpha}_2$ (Exercise), and $\bar{\bar{\alpha}}=\alpha$ (flip twice). It also yields formulas for the real and imaginary parts:

$$\alpha+\bar{\alpha}=(a+b\mathbf{i})+(a-b\mathbf{i})=2a\implies\mathrm{Re}(\alpha)=\frac{\alpha+\bar{\alpha}}{2}\tag{I.A.4}$$

$$\alpha-\bar{\alpha}=(a+b\mathbf{i})-(a-b\mathbf{i})=2b\mathbf{i}\implies\mathrm{Im}(\alpha)=\frac{\alpha-\bar{\alpha}}{2\mathbf{i}}.\tag{I.A.5}$$

I.A.6. DEFINITION. The **modulus** (or **norm** or **absolute value**) of a complex number α $(= a + b\mathbf{i})$ is

$$|\alpha| := \sqrt{a^2 + b^2}.$$

It may be thought of as the distance from 0 to α, or the length of the corresponding vector.

We may relate conjugation, modulus and inverse as follows:

$$\begin{aligned} \alpha\overline{\alpha} &= (a+b\mathbf{i})(a-b\mathbf{i}) = a^2+b^2+(ab-ab)\mathbf{i} = |\alpha|^2 \\ &\implies \alpha \cdot \frac{\overline{\alpha}}{|\alpha|^2} = 1 \implies \alpha^{-1} = \frac{\overline{\alpha}}{|\alpha|^2}. \end{aligned} \tag{I.A.7}$$

When dividing by a complex number in practice, it's easier to write

$$\frac{a+b\mathbf{i}}{c+d\mathbf{i}} = \frac{(a+b\mathbf{i})(c-d\mathbf{i})}{(c+d\mathbf{i})(c-d\mathbf{i})} = \frac{(ac+bd)+(bc-ad)\mathbf{i}}{c^2+d^2}.$$

I.A.8. PROPERTIES. The modulus satisfies the following:

- $|\overline{\alpha}| = |\alpha|$
- $|\alpha\beta| = |\alpha||\beta|$ (Check: $|\alpha\beta|^2 = (\alpha\beta)(\overline{\alpha\beta}) = (\alpha\overline{\alpha})(\beta\overline{\beta}) = |\alpha|^2|\beta|^2$)
- triangle inequality: $|\alpha+\beta| \leq |\alpha| + |\beta|$
- immediate consequences: $|\sum \alpha_i| \leq \sum |\alpha_i|$ by induction; and writing $\alpha = (\alpha - \beta) + \beta$ yields $|\alpha - \beta| \geq |\alpha| - |\beta|$.

PROOF OF TRIANGLE INEQUALITY. For $z = x + \mathbf{i}y \in \mathbb{C}$,

$$\mathrm{Re}(z) = x \leq |x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} = |z|.$$

Now $|\alpha+\beta|^2$

$$\begin{aligned} |\alpha+\beta|^2 &= (\alpha+\beta)\overline{(\alpha+\beta)} = \alpha\overline{\alpha} + \beta\overline{\beta} + \alpha\overline{\beta} + \overline{\alpha}\beta \\ &= |\alpha|^2 + |\beta|^2 + 2\mathrm{Re}(\alpha\overline{\beta}) \\ &\leq |\alpha|^2 + |\beta|^2 + 2|\alpha\overline{\beta}| = |\alpha|^2 + |\beta|^2 + 2|\alpha||\overline{\beta}| \\ &= |\alpha|^2 + |\beta|^2 + 2|\alpha||\beta| = (|\alpha| + |\beta|)^2. \end{aligned}$$

Taking square roots gives the result. $\square$

Cauchy-Schwarz inequality. Define a "Hermitian inner product" on $\mathbb{C}^n$ (= complex n-vectors) by

$$\langle \underline{\alpha}, \underline{\beta} \rangle := \underline{\alpha} \cdot \overline{\underline{\beta}} := \sum_{i=1}^{n} \alpha_i \overline{\beta}_i ,$$

where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\underline{\beta} = (\beta_1, \ldots, \beta_n)$. Write $\|\underline{\alpha}\| := \sqrt{\langle \underline{\alpha}, \underline{\alpha} \rangle}$ for the norm; it is always ≥ 0, with equality iff $\underline{\alpha} = \underline{0}$. Note that $\overline{\langle \underline{\alpha}, \underline{\beta} \rangle} = \langle \underline{\beta}, \underline{\alpha} \rangle$.

I.A.9. PROPOSITION. *We have* $|\langle \underline{\alpha}, \underline{\beta} \rangle| \leq \|\underline{\alpha}\| \|\underline{\beta}\|$, *with equality if and only if* $\underline{\alpha} = \gamma \underline{\beta}$ *for some* $\gamma \in \mathbb{C}$.

PROOF. Let $\underline{\alpha}, \underline{\beta} \in \mathbb{C}^n$ and compute:

$$0 \leq \left\| \|\underline{\beta}\| \underline{\alpha} - \frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\beta}\|} \underline{\beta} \right\|^2 = \left\langle \|\underline{\beta}\| \underline{\alpha} - \frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\beta}\|} \underline{\beta},\ \|\underline{\beta}\| \underline{\alpha} - \frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\beta}\|} \underline{\beta} \right\rangle$$

$$= \|\underline{\beta}\|^2 \|\underline{\alpha}\|^2 - \langle \underline{\alpha}, \underline{\beta} \rangle \langle \underline{\beta}, \underline{\alpha} \rangle - \langle \underline{\alpha}, \underline{\beta} \rangle \overline{\langle \underline{\alpha}, \underline{\beta} \rangle} + \frac{|\langle \underline{\alpha}, \underline{\beta} \rangle|^2 \cancel{\langle \underline{\beta}, \underline{\beta} \rangle}}{\cancel{\|\underline{\beta}\|^2}}$$

which after cancelling the last two terms and rewriting $\langle \underline{\beta}, \underline{\alpha} \rangle$ as $\overline{\langle \underline{\alpha}, \underline{\beta} \rangle}$ becomes

$$= \|\underline{\beta}\|^2 \|\underline{\alpha}\|^2 - |\langle \underline{\alpha}, \underline{\beta} \rangle|^2,$$

whence the inequality. $\square$

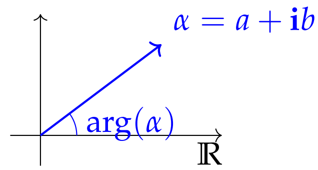
I.A.10. REMARK. If $\underline{\alpha}, \underline{\beta} \in \mathbb{R}^n$, then

$$\arccos \left(\frac{\langle \underline{\alpha}, \underline{\beta} \rangle}{\|\underline{\alpha}\| \|\underline{\beta}\|} \right)$$

gives the usual Euclidean angle between them.

§ **Polar form.**

The argument $\arg(\alpha)$ of a complex number is the angle it makes with the $\mathbb{R}_{\geq 0}$-axis



which of course is $\arctan(\frac{b}{a})$. We may then write

$$\alpha = |\alpha|(\cos(\arg(\alpha)) + \mathbf{i}\sin(\arg(\alpha)))$$

in polar form. Note that

- the parenthetical expression has modulus 1 (since $\cos^2 + \sin^2 = 1$)
- $\arg(\alpha)$ is defined mod 2π (or "mod $2\pi\mathbb{Z}$").

I.A.11. CLAIM. $\arg(\alpha_1\alpha_2) \equiv \arg(\alpha_1) + \arg(\alpha_2) \pmod{2\pi}$.

PROOF. Define $e^{\mathbf{i}\theta} := \cos\theta + \mathbf{i}\sin\theta$ for $\theta \in \mathbb{R}$. Then

$$\begin{aligned} e^{\mathbf{i}\theta}e^{\mathbf{i}\phi} &= (\cos\theta + \mathbf{i}\sin\theta)(\cos\phi + \mathbf{i}\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + \mathbf{i}(\sin\theta\cos\phi + \cos\theta\sin\phi) \\ &= \cos(\theta+\phi) + \mathbf{i}\sin(\theta+\phi) \\ &= e^{\mathbf{i}(\theta+\phi)}. \end{aligned}$$

In particular, $(e^{\mathbf{i}\phi})^{-1} = e^{-\mathbf{i}\phi}$ $(= \overline{e^{\mathbf{i}\phi}})$ since $e^{\mathbf{i}\phi}e^{-\mathbf{i}\phi} = e^{\mathbf{i}(\phi-\phi)} = e^{\mathbf{i}0} = 1$. So

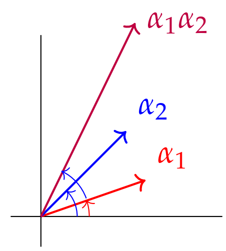
$$\begin{aligned} e^{\mathbf{i}\theta} = e^{\mathbf{i}\phi} &\iff e^{\mathbf{i}(\theta-\phi)} = 1 \\ &\iff \cos(\theta-\phi) = 1 \\ &\iff \theta \equiv \phi \pmod{2\pi}, \end{aligned}$$

and

$$\begin{aligned} |\alpha_1\alpha_2|e^{\mathbf{i}(\arg\alpha_1 + \arg\alpha_2)} &= (|\alpha_1|e^{\mathbf{i}\arg\alpha_1})(|\alpha_2|e^{\mathbf{i}\arg\alpha_2}) \\ &= \alpha_1\alpha_2 \\ &= |\alpha_1\alpha_2|e^{\mathbf{i}\arg(\alpha_1\alpha_2)} \end{aligned}$$

$\implies e^{\mathbf{i}(\arg\alpha_1+\arg\alpha_2)} = e^{\mathbf{i}\arg(\alpha_1\alpha_2)} \implies \arg(\alpha_1)+\arg(\alpha_2) \equiv \arg(\alpha_1\alpha_2)$ modulo 2π. □

Hence multiplying complex numbers adds angles and multiplies lengths:



Simplifying notation, we will write $\alpha_j = r_j e^{\mathbf{i}\theta_j}$. Quotients then look like

$$\frac{\alpha_1}{\alpha_2} = \frac{r_1}{r_2} e^{\mathbf{i}(\theta_1-\theta_2)}.$$

Roots. Solve $z^n = \alpha$, for $\alpha = re^{\mathbf{i}\theta}$; i.e. find $\alpha^{\frac{1}{n}}$. *One* solution is given by $z = \sqrt[n]{r}\, e^{\mathbf{i}(\theta/n)}$ (but see the Exercises).

I.A.12. EXAMPLE ($n = 2$). For $\theta \in [0, \pi]$, we have

$$e^{\mathbf{i}(\theta/2)} = \cos(\theta/2) + \mathbf{i}\sin(\theta/2) = \sqrt{\frac{1+\cos\theta}{2}} + \mathbf{i}\sqrt{\frac{1-\cos\theta}{2}}.$$

Conversions.

- polar $\to$ Cartesian: $re^{\mathbf{i}\theta} = r\cos\theta + \mathbf{i}r\sin\theta$
- polar $\leftarrow$ Cartesian: $a + \mathbf{i}b = \sqrt{a^2+b^2}\, e^{\mathbf{i}\arctan(b/a)}$

So for $\alpha = a + b\mathbf{i}$ with $b \geq 0$, one square root is given by

$$\begin{aligned}
\alpha^{\frac{1}{2}} &= r^{\frac{1}{2}} e^{\mathbf{i}\theta/2} \\
&= r^{\frac{1}{2}}\sqrt{\frac{1+\cos\theta}{2}} + \mathbf{i}r^{\frac{1}{2}}\sqrt{\frac{1-\cos\theta}{2}} \\
&= \sqrt{\frac{r+r\cos\theta}{2}} + \mathbf{i}\sqrt{\frac{r-r\cos\theta}{2}} \\
&= \sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + \mathbf{i}\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}.
\end{aligned}$$

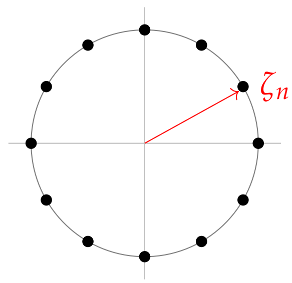
If $b < 0$, then we change the sign of the imaginary term.

I.A.13. EXAMPLE. $\sqrt{3+4\mathbf{i}} = \sqrt{\frac{5+3}{2}} + \mathbf{i}\sqrt{\frac{5-3}{2}} = 2+\mathbf{i}$.
(Check: $(2+\mathbf{i})(2+\mathbf{i}) = (4-1)+\mathbf{i}(2+2) = 3+4\mathbf{i}$.)

Roots of unity. The equation $z^n = 1$ has n solutions

$$1,\ \zeta_n,\ \zeta_n^2,\ \ldots,\ \zeta_n^{n-1}$$

where $\zeta_n = e^{\mathbf{i}\frac{2\pi}{n}}$.



We can also write them as $e^{\mathbf{i}\frac{2\pi k}{n}}$ where $k = 0, 1, \ldots, n-1$. Note that $\zeta_n^{n-1}\zeta_n = \zeta_n^n = 1 \implies \zeta_n^{n-1} = \zeta_n^{-1} = \overline{\zeta_n}$.

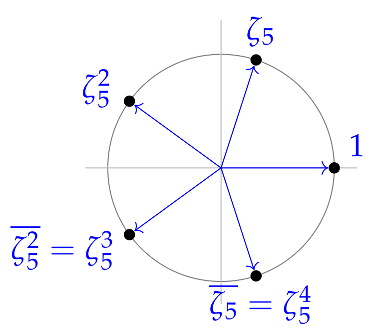
I.A.14. EXAMPLES.

- $n = 2 : 1, -1$
- $n = 3 : 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$
- $n = 4 : 1, -1, \mathbf{i}, -\mathbf{i}$

After $n = 4$, they get harder to determine in Cartesian form. For

- $n = 5$

we may proceed as follows:



Writing $\zeta_5 = a + b\mathbf{i}$ with $a^2 + b^2 = 1$, we have

$$\zeta_5^2 = (a^2 - b^2) + 2ab\mathbf{i} = (2a^2 - 1) + 2ab\mathbf{i}$$

and thus

$$\zeta_5^3 = (\zeta_5^2)\zeta_5 = (\cdots\cdots) + (2a^2b + (2a^2-1)b)\mathbf{i}.$$

Now

$$\begin{aligned}(\zeta_5^2)\zeta_5 = \zeta_5^3 = \overline{\zeta_5^2} &\Longrightarrow \mathrm{Im}[(\zeta_5^2)\zeta_5] = \mathrm{Im}(\overline{\zeta_5^2}) \\ &\Longrightarrow 2a^2b + (2a^2-1)b = -2ab \\ &\Longrightarrow 4a^2 + 2a - 1 = 0 \\ &\Longrightarrow \begin{cases} a = \frac{-2+\sqrt{4+16}}{8} = \frac{-1+\sqrt{5}}{4} \text{ and} \\ b = \sqrt{1-a^2} = \sqrt{1-\left(\frac{\sqrt{5}-1}{4}\right)^2} = \sqrt{\frac{5+\sqrt{5}}{8}}. \end{cases}\end{aligned}$$

Hence,

$$\zeta_5 = \frac{-1+\sqrt{5}}{4} + \mathbf{i}\sqrt{\frac{5+\sqrt{5}}{8}}. \tag{I.A.15}$$

(These are cos and sin of $\frac{360^\circ}{5} = 72^\circ$)

Finally, recalling that

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha}$$

we have

$$1 + \zeta_n + \zeta_n^2 + \cdots + \zeta_n^{n-1} = \frac{1-\zeta_n^n}{1-\zeta_n} = \frac{0}{1-\zeta_n} = 0. \tag{I.A.16}$$

§ Spherical representation (the Riemann sphere).

For many reasons — topological, geometric, analytic — it is convenient to "compactify" (or "projectively complete") $\mathbb{C}$ by adding a "point at ∞". Writing[2]

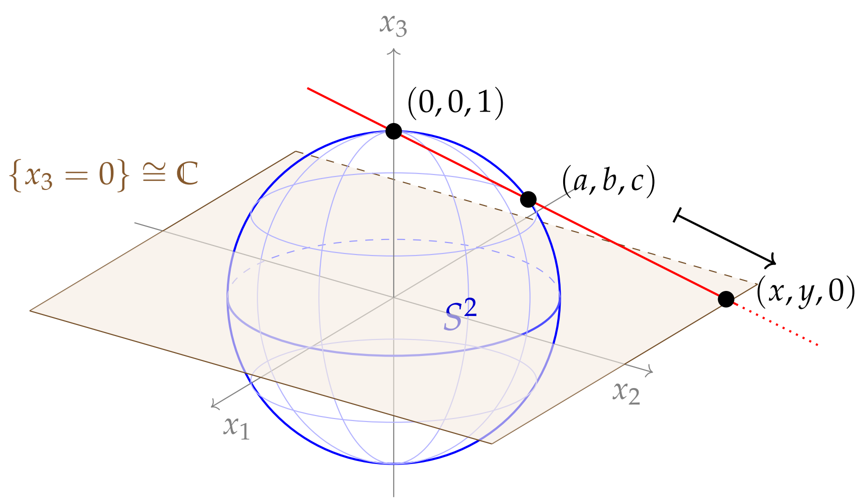
$$\hat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\},$$

we topologize the result by defining basic open neighborhoods of ∞ to be sets of the form

$$\{z \in \mathbb{C} : |z| > R\} \sqcup \{\infty\},$$

and letting $\mathbb{C}$ have the usual Euclidean topology.[3]

To get a geometric model for $\hat{\mathbb{C}}$, consider the picture in $\mathbb{R}^3$



where (a,b,c) is a point on the unit sphere $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$. Define the **stereographic projection** map

$$\pi\colon S^2 \to \hat{\mathbb{C}} \tag{I.A.17}$$

to send $(0,0,1)$ to ∞ and any other (a,b,c) to $L_{(a,b,c),(0,0,1)} \cap \{x_3 = 0\}$ $= (x,y,0)$ as shown.

[2]Here $\sqcup$ denotes disjoint union of sets.

[3]Another way to think of this is as $(\mathbb{C}^2 \setminus \{\underline{0}\})/\mathbb{C}^*$, where the quotient means that for any $\alpha \in \mathbb{C}^*$ we declare (z_1, z_2) and $(\alpha z_1, \alpha z_2)$ equivalent. The result of this construction is called $\mathbb{P}^1$, the **complex projective line** (but it's the same as $\hat{\mathbb{C}}$).

We can explicitly calculate the image point $(x, y, 0)$ as follows. First parametrize the line L by $t \mapsto (at, bt, 1 + (c-1)t)$. Solving $1 + (c-1)t = 0$ gives $t = \frac{1}{1-c}$, and substituting gives $((x, y, 0) =)$ $(\frac{a}{1-c}, \frac{b}{1-c}, 0)$. As a point of $\mathbb{C}$, this is $z = \frac{a}{1-c} + \mathbf{i}\frac{b}{1-c}$. Thus we can define (I.A.17) more explicitly by

$$\pi(a, b, c) := \frac{a + \mathbf{i}b}{1-c}. \tag{I.A.18}$$

To find the inverse map, use $a^2 + b^2 + c^2 = 1$ to write

$$|z| = \frac{\sqrt{a^2 + b^2}}{|1-c|} = \frac{\sqrt{1-c^2}}{|1-c|} = \sqrt{\frac{1+c}{1-c}}$$

This implies:

(1) that "level curves" $\{c = \text{constant}\} \subset S^2$ map to circles $\{|z| = \text{constant}\}$ centered about $0 \in \mathbb{C}$;
(2) that the upper hemisphere $(c > 0)$ of S^2 maps to $|z| > 1$ and the lower hemisphere $(c < 0)$ of S^2 maps to $|z| < 1$;
(3) $|z|^2 = \frac{1+c}{1-c} \implies |z|^2 - |z|^2 c = 1 + c \implies \frac{|z|^2-1}{|z|^2+1} = c$

$$\begin{cases} a = (1-c)\mathrm{Re}(z) = \frac{2}{|z|^2+1} \cdot \frac{z+\bar{z}}{2} = \frac{z+\bar{z}}{|z|^2+1} \\ b = (1-c)\mathrm{Im}(z) = \frac{2}{|z|^2+1} \cdot \frac{z-\bar{z}}{2\mathbf{i}} = \frac{z-\bar{z}}{\mathbf{i}(|z|^2+1)} \end{cases}$$

(which gives an explicit inverse); and we conclude
(4) that π is 1-to-1.

(1) and (4) essentially yield part (a) of

I.A.19. THEOREM. *(a) π is a homeomorphism (= 1-to-1 continuous map with continuous inverse map) between S^2 and $\hat{\mathbb{C}}$. Either one is called the* ***Riemann sphere****.*

(b) Circles [resp. lines] in $\mathbb{C}$ correspond (under π) to circles on S^2 not through $(0, 0, 1)$ [resp. passing through $(0, 0, 1)$].

SKETCH OF (b). For the lines, there are 1-to-1 correspondences from planes through $(0, 0, 1)$ not parallel to $\{x_3 = 0\}$ to:

- lines in $\{x_3 = 0\}$ (by intersecting with $\{x_3 = 0\}$); and
- circles on S^2 passing through $(0, 0, 1)$ (by intersecting with S^2).

For the circles, start with $C \subset S^2$. A circle lives in a plane, so must be of the form

$$C = S^2 \cap \{a_1 x_1 + a_2 x_2 + a_3 x_3 = a_0\}$$

whose image under π is

$$\left\{ a_1 \frac{z + \overline{z}}{|z|^2 + 1} + a_2 \frac{z - \overline{z}}{\mathbf{i}(|z|^2 + 1)} + a_3 \frac{|z|^2 - 1}{|z|^2 + 1} = a_0 \right\} \subseteq \mathbb{C}.$$

Writing $z = x + \mathbf{i}y$, this is (after multiplying through by $|z|^2 + 1$)

$$2a_1 x + 2a_2 y + a_3(x^2 + y^2 - 1) = a_0(x^2 + y^2 + 1)$$

or

$$0 = (a_0 - a_3)(x^2 + y^2) - 2a_1 x - 2a_2 y + (a_0 + a_3),$$

the equation of a circle.[4] Conversely, given any equation of the form $0 = A(x^2 + y^2) + Bx + Cy + D$, we can solve for a_0, a_1, a_2, a_3 (non-uniquely), hence realize its solution set as $\pi(\text{circle})$. $\square$

Exercises

(1) If $z = x + \mathbf{i}y$ ($x, y \in \mathbb{R}$), find Re and Im of $\frac{z-1}{z+1}$.
(2) Show that the system of all 2×2 real matrices of the special form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ is isomorphic to the field of complex numbers.
(3) Prove the Cauchy-Schwarz inequality by induction.
(4) Let $\pi \colon S^2 \to \hat{\mathbb{C}}$ denote stereographic projection. Show that the map $(a, b, c) \overset{\imath}{\mapsto} (-a, -b, -c)$ on the 2-sphere S^2 corresponds (under π) to the map $z \overset{\jmath}{\mapsto} \frac{-1}{\overline{z}}$ on $\hat{\mathbb{C}}$. Deduce that $\jmath$ (and hence also $z \mapsto \frac{-1}{z}$) sends circles in $\mathbb{C}^*$ to circles in $\mathbb{C}^*$. (Note: $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.)
(5) Describe geometrically the set of points $z \in \mathbb{C}$ satisfying $|z - \mathbf{i} + 3| \leq 5$.
(6) Put (a) $1 + \mathbf{i}\sqrt{2}$ in polar form, and (b) $e^{-5\mathbf{i}\pi/4}$ in Cartesian form.
(7) Find the square roots of $\sqrt{3} + 3\mathbf{i}$ (in Cartesian form).
(8) Show that complex conjugation respects multiplication, $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$.

[4]Recall $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$ is the equation of a general conic. This yields a circle (or a point or the empty set, which we needn't worry about) if $C = 0$ and $A = B$.