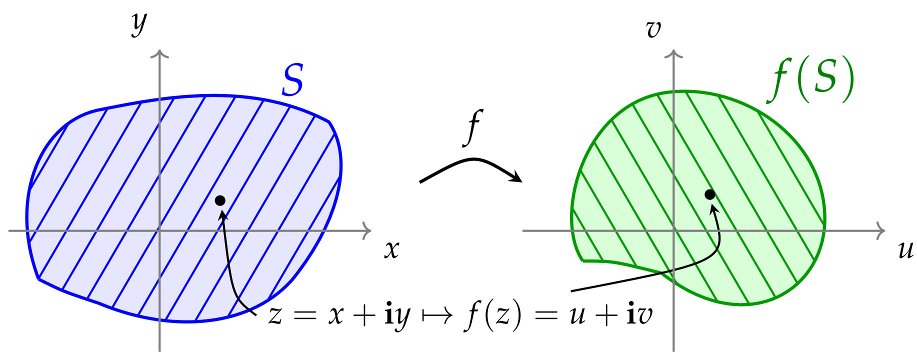
I.B. Complex functions

§ Some examples.

We will consider complex-valued functions

$$f\colon S \to \mathbb{C}$$

on a subset $S \subseteq \mathbb{C}$.



The interplay between "$f\colon \mathbb{C} \to \mathbb{C}$" and "$F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$" is especially important here. To be able to discuss this clearly, we shall fix the notation

(I.B.1) $$f(x+\mathbf{i}y) = f(z) = u(z) + \mathbf{i}v(z) = u(x,y) + \mathbf{i}v(x,y)$$

for the duration of Chapter I.B.

I.B.2. EXAMPLE (Exponential map). Define

$$f(z) = e^z := e^x e^{\mathbf{i}y} = \underbrace{e^x \cos y}_{u(x,y)} + \mathbf{i}\,\underbrace{e^x \sin y}_{v(x,y)}.$$

If

$$S = \{x + \mathbf{i}y \mid x \geq c,\ 0 \leq y \leq 2\pi\} \subseteq \mathbb{C},$$

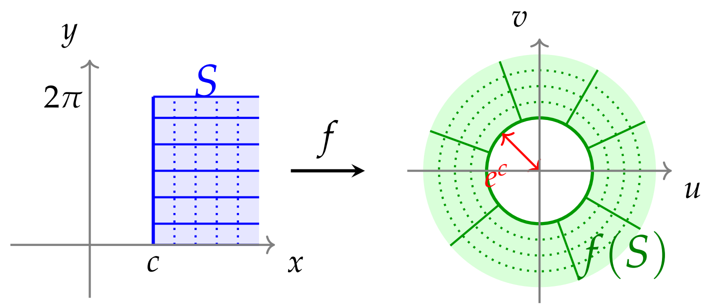
then

$$f(S) = \{w \in \mathbb{C} \mid |w| \geq e^c\},$$

as

$$|e^z| = e^x \geq e^c$$

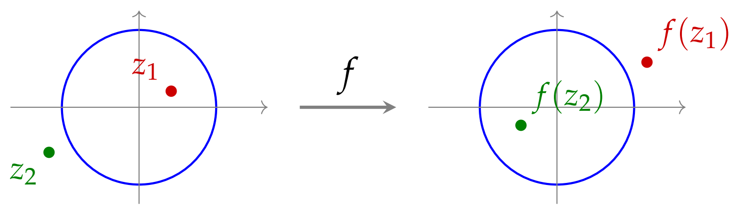
and $e^{\mathbf{i}y}$ takes all angles when $y \in [0, 2\pi]$.



I.B.3. EXAMPLE (Inversion about unit circle).

$$f(z) = \frac{1}{\bar{z}} = \frac{1}{x - \mathbf{i}y} = \frac{x + \mathbf{i}y}{x^2 + y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} + \mathbf{i} \underbrace{\frac{y}{x^2+y^2}}_{v(x,y)}$$

$$= \frac{1}{r}\left(\frac{x}{r} + \mathbf{i}\frac{y}{r}\right) = \frac{1}{r}(\cos(\theta) + \mathbf{i}\sin(\theta)) = \frac{1}{r}e^{\mathbf{i}\theta}$$



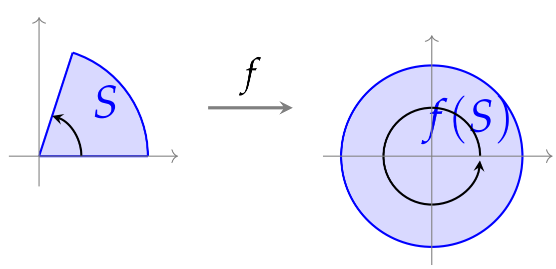
The transformation "preserves angle but inverts length".

I.B.4. EXAMPLE.

$$f(z) = z^5 = r^5 e^{\mathbf{i}(5\theta)}$$

$$u(z) = r^5 \cos(5\theta), \quad v(z) = r^5 \sin(5\theta)$$

(This is less easy to express in terms of x and y.)



§ Complex differentiability.

For a real function of a real variable $g\colon I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, recall that the properties of being

(a) differentiable,
(b) infinitely differentiable ("smooth" or C^∞), and
(c) real analytic (locally representable by power series)

are all distinct.

I.B.5. EXAMPLE.

$$g(x) = \begin{cases} 0 & x \leq 0 \\ e^{-1/x^2} & x > 0 \end{cases}$$

is smooth but *non-analytic*: all derivatives are zero at $x = 0$, and so g cannot be described by a power series there.

Now let U be an open subset of $\mathbb{C}$. For a complex function

$$f\colon U \to \mathbb{C},$$

the properties (to be defined) of being

(a) complex differentiable ("holomorphic"),
(b) infinitely complex differentiable,[5] and
(c) complex analytic (power series in complex variable)

will turn out, in contrast, to be equivalent!

Let's start from a multivariable (real) calculus perspective: suppose we are given $\vec{F}\colon U \to \mathbb{R}^2$, $U \subseteq \mathbb{R}^2$ open, and write

$$\vec{F}(\vec{z}) = \vec{F}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right).$$

This $\vec{F}$ is **differentiable** (in the real 2-variable $\leftrightarrow$ 2-variable sense) at $\vec{z}_0 \in U$ if there is a matrix $A \in M_2(\mathbb{R})$ such that

$$\lim_{\vec{h}\to\vec{0}} \frac{\|\vec{F}(\vec{z}_0 + \vec{h}) - \vec{F}(\vec{z}_0) - A\vec{h}\|}{\|\vec{h}\|} = 0. \tag{I.B.6}$$

[5] *not* called smooth: that term refers exclusively to infinite *real* differentiability.

Now (I.B.6) implies existence of the 4 partials u_x, u_y, v_x, v_y at $\vec{z}_0$, and in fact

$$\text{(I.B.7)} \qquad A = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix} =: J_{\vec{F}}(\vec{z}_0)$$

is the **Jacobian matrix**.

A partial converse is given by the

I.B.8. PROPOSITION. *If the 4 partials exist and are continuous on U, then* (I.B.6) *holds (with $A = J_{\vec{F}}(\vec{z}_0)$) for all $\vec{z}_0 \in U$.*

PROOF. Without loss of generality, we may assume that $\vec{z}_0 = \vec{0} = \vec{F}(\vec{z}_0)$. Using the triangle inequality, are taking $\vec{h} = \binom{\Delta x}{\Delta y}$, we have

$$\begin{aligned} \|\vec{F}(\vec{h}) - J_{\vec{F}}(\vec{0})\cdot\vec{h}\| \;\leq\; & \left\| \vec{F}\binom{\Delta x}{\Delta y} - \vec{F}\binom{\Delta x}{0} - J_{\vec{F}}\binom{\Delta x}{0}\cdot\binom{0}{\Delta y} \right\| \quad \textbf{(i)} \\ & + \left\| \vec{F}\binom{\Delta x}{0} - \vec{F}\binom{0}{0} - J_{\vec{F}}\binom{0}{0}\cdot\binom{\Delta x}{0} \right\| \quad \textbf{(ii)} \\ & + \left\| J_{\vec{F}}\binom{\Delta x}{0}\cdot\binom{0}{\Delta y} + J_{\vec{F}}\binom{0}{0}\cdot\binom{\Delta x}{0} - J_{\vec{F}}\binom{0}{0}\cdot\binom{\Delta x}{\Delta y} \right\| \quad \textbf{(iii)} \end{aligned}$$

Now expanding

$$\textbf{(i)} = \left\| \begin{pmatrix} u\binom{\Delta x}{\Delta y} - u\binom{\Delta x}{0} - u_y\binom{\Delta x}{0}\Delta y \\ v\binom{\Delta x}{\Delta y} - v\binom{\Delta x}{0} - v_y\binom{\Delta x}{0}\Delta y \end{pmatrix} \right\|,$$

each entry of the vector divided by Δy goes to zero (with Δy, *a fortiori* with $\|\vec{h}\|$) by the definition of partial derivative. But then $\frac{\textbf{(i)}}{\|\vec{h}\|} \to 0$, since $\frac{|\Delta y|}{\|\vec{h}\|} \leq 1$.

A similar argument yields $\frac{\textbf{(ii)}}{\|\vec{h}\|} \to 0$. Finally,

$$\textbf{(iii)} = \left\| \begin{pmatrix} u_y\binom{\Delta x}{0}\Delta y + \cancel{u_x\binom{0}{0}\Delta x} - [\cancel{u_x\binom{0}{0}\Delta x} + u_y\binom{0}{0}\Delta y] \\ v_y\binom{\Delta x}{0}\Delta y + \cancel{v_x\binom{0}{0}\Delta x} - [\cancel{v_x\binom{0}{0}\Delta x} + v_y\binom{0}{0}\Delta y] \end{pmatrix} \right\|$$

$$\text{whence} \qquad \frac{\textbf{(iii)}}{\|\vec{h}\|} = \frac{|\Delta y|}{\|\vec{h}\|} \cdot \left\| \begin{pmatrix} u_y\binom{\Delta x}{0} - u_y\binom{0}{0} \\ v_y\binom{\Delta x}{0} - v_y\binom{0}{0} \end{pmatrix} \right\| \to 0$$

by the continuity of u_y and v_y. $\square$

Now call $f\colon U \to \mathbb{C}$ **complex differentiable** at $z_0 \in U$ if there exists $\alpha \in \mathbb{C}$ such that

$$\lim_{h\to 0} \frac{|f(z_0+h) - f(z_0) - \alpha h|}{|h|} = 0.$$

In this case, we shall denote $\alpha =: f'(z_0) = \left.\frac{df}{dz}\right|_{z_0}$.

I.B.9. DEFINITION. f is **holomorphic on** $U \iff f$ is complex differentiable at every point of U.

I.B.10. WARNING. I don't consider this to be a correct definition of (complex) "analytic", even though they will turn out to be the same. This is my only serious terminological difference with Ahlfors. (Moreover, it's clear that in both cases differentiability $\implies$ continuity.)

We now want to compare the two notions of differentiability, for $\vec{F}$ and f. Writing $\vec{F} = \left(\begin{smallmatrix} u(x,y) \\ v(x,y) \end{smallmatrix}\right)$ and $f = u(x,y) + \mathbf{i}v(x,y)$, first we observe that

- f is continuous (at z_0) $\iff \lim_{h\to 0} f(z_0+h) = f(z_0)$
- $\vec{F}$ is continuous (at $\vec{z}_0$) $\iff \lim_{\vec{h}\to\vec{0}} \vec{F}(\vec{z}_0+\vec{h}) = \vec{F}(\vec{z}_0)$.

These are clearly the same thing. Moreover, it's clear in both cases that differentiability implies continuity.

Let's write out complex differentiability in matrix/vector form: with $\alpha = a + \mathbf{i}b$, $h = \Delta x + \mathbf{i}\Delta y$, and $f = u + \mathbf{i}v$, we have

(I.B.11)

$$\begin{aligned} \lim_{h\to 0} \frac{|f(z_0+h) - f(z_0) - \alpha h|}{|h|} &= \lim_{\vec{h}\to\vec{0}} \frac{\left\| \vec{F}(\vec{z}_0+\vec{h}) - \vec{F}(\vec{z}_0) - \begin{pmatrix} \mathrm{Re}(\alpha h) \\ \mathrm{Im}(\alpha h) \end{pmatrix} \right\|}{\|\vec{h}\|} \\ &= \lim_{\vec{h}\to\vec{0}} \frac{\left\| \vec{F}(\vec{z}_0+\vec{h}) - \vec{F}(\vec{z}_0) - \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \vec{h} \right\|}{\|\vec{h}\|}, \end{aligned}$$

where the second equality comes from the key computation:

$$\begin{aligned} \alpha h = (a+\mathbf{i}b)(\Delta x + \mathbf{i}\Delta y) &= (a\Delta x - b\Delta y) + \mathbf{i}(b\Delta x + a\Delta y) \\ &\implies \begin{pmatrix} \mathrm{Re}(\alpha h) \\ \mathrm{Im}(\alpha h) \end{pmatrix} = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}. \end{aligned}$$

Notice that the Jacobian matrix in (I.B.11) takes the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{I.B.12}$$

which suggests that complex differentiability (of f) is more "rigidifying" than real differentiability (of $\vec{F}$). The next result makes this more precise.

I.B.13. THEOREM. (a) *Suppose the four partials u_x, u_y, v_x, v_y exist and are continuous on U. If in addition they satisfy the* **Cauchy-Riemann equations**

$$u_x = v_y \quad \text{and} \quad u_y = -v_x, \tag{I.B.14}$$

then f is holomorphic on U.

(b) *If f is holomorphic on U, then the four partials exist everywhere and satisfy the Cauchy-Riemann equations.*

PROOF. (b) f complex differentiable $\implies$ LHS (I.B.11) $= 0 \implies$ RHS (I.B.11) $= 0 \implies \vec{F}$ differentiable with $\left(\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right) = J_{\vec{F}}$ everywhere of the form $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right) \implies$ Cauchy-Riemann equations hold.

(a) The existence and continuity of partials $\implies \vec{F}$ differentiable (use Prop. I.B.8). Then the C-R equations $\implies J_{\vec{F}} = \left(\begin{smallmatrix} u_x & -v_x \\ v_x & u_x \end{smallmatrix}\right) \implies$ RHS (I.B.11) $= 0 \implies$ LHS (I.B.11) $= 0 \implies f$ holomorphic. $\square$

ALTERNATE PROOF OF (b). Here is a considerably more intuitive approach. The definition of complex differentiability says that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} =: f'(z_0)$$

exists. In particular, *it is independent of the direction by which $h \to 0$.* That is, at each point, taking the limit along the real axis ($h = \Delta x$)

$$\lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \left(\frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + \mathbf{i} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right)$$

$$= \frac{\partial u}{\partial x} + \mathbf{i} \frac{\partial v}{\partial x}$$

and along the imaginary axis ($h = \mathbf{i}\Delta y$)

$$\lim_{\Delta y \to 0} \frac{f(z + \mathbf{i}\Delta y) - f(z)}{\mathbf{i}\Delta y} =$$

$$\lim_{\Delta y \to 0} \left(\frac{u(x, y + \Delta y) - u(x, y)}{\mathbf{i}\Delta y} + \mathbf{i}\frac{v(x, y + \Delta y) - v(x, y)}{\mathbf{i}\Delta y} \right)$$

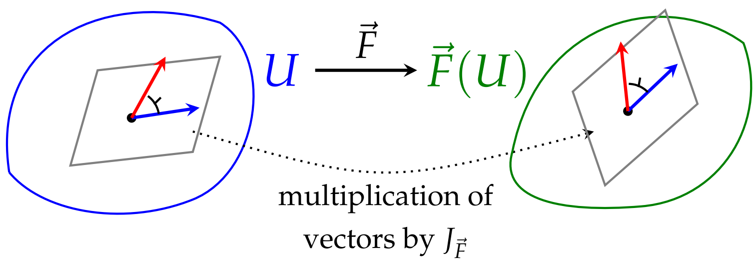
$$= -\mathbf{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

must yield the same result. Equating real and imaginary parts gives $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$. □

I.B.15. REMARK. Using $|\alpha| = \sqrt{a^2 + b^2}$ and $\varphi = \arg(\alpha)$, the polar form $\alpha = |\alpha| e^{\mathbf{i}\varphi}$ has a related matrix factorization

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} |\alpha| & 0 \\ 0 & |\alpha| \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

into a dilation times a rotation. These operations preserve angles.



Now suppose we are in the complex differentiable setting, so that $J_{\vec{F}}$ takes the form (I.B.12) at every point of U. Since the Jacobian at $\vec{z}_0$ expresses the *infinitesimal* linear transformation on vectors (in the tangent plane) at $\vec{z}_0$, the oriented angles between these vectors *do not change* in the image. The "integrated" form of this statement, for a complex differentiable f *with nowhere vanishing derivative*, is that f doesn't change the angle between (the tangent vectors of) curves where they meet. In this case we say that f is **conformal**.

Note that the polar form expresses any complex number as a rotation $e^{\mathbf{i}\theta}$ times a dilation r. Exercise (2) from I.A makes the correspondence between matrices and complex numbers a bit more formal.

§ The operators $\partial/\partial z$ and $\partial/\partial\bar{z}$.

We now turn to an alternate form of the Cauchy-Riemann equations. Consider the partial differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathbf{i}\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right). \tag{I.B.16}$$

We shall apply them to functions $f\colon U \to \mathbb{C}$ which may or may not be complex differentiable. However we do assume that the partials $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ exist and are continuous — that is, "f is continuously $\mathbb{R}$-differentiable".

If f is holomorphic, then

$$\begin{aligned}\frac{\partial}{\partial z}f &= \frac{1}{2}\left\{\frac{\partial}{\partial x}(u+\mathbf{i}v) - \mathbf{i}\frac{\partial}{\partial y}(u+\mathbf{i}v)\right\}\\ &= \frac{1}{2}\left\{\frac{\partial u}{\partial x} + \mathbf{i}\frac{\partial v}{\partial x} + \mathbf{i}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right\} \quad \text{(by the C-R eqns)}\\ &= \frac{\partial u}{\partial x} + \mathbf{i}\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u+\mathbf{i}v) = \frac{\partial f}{\partial x}\\ &= \frac{df}{dz},\end{aligned} \tag{I.B.17}$$

where the last equality is by "independence of direction" of $h \to 0$ in the definition of the complex derivative.[6]

By a similar argument, one has (*if* f is holomorphic)

$$\frac{\partial}{\partial \bar{z}}f = \frac{1}{2}\left\{\frac{\partial}{\partial x}(u+\mathbf{i}v) + \mathbf{i}\frac{\partial}{\partial y}(u+\mathbf{i}v)\right\} = 0. \tag{I.B.18}$$

So we have (see the Exercises):

$$f \text{ holomorphic} \iff \frac{\partial f}{\partial z} = \frac{df}{dz} \iff \frac{\partial f}{\partial \bar{z}} = 0. \tag{I.B.19}$$

Regardless of holomorphicity of f, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ obey the usual sum, product, and quotient rules (as $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ do).

[6]Taking $h \to 0$ along the real axis gives $\frac{df}{dz} = \frac{\partial f}{\partial x}$ for holomorphic f; taking it along the imaginary axis gives $\frac{df}{dz} = \frac{\partial f}{\partial(\mathbf{i}y)} = -\mathbf{i}\frac{\partial f}{\partial y}$ for holomorphic f. This is essentially how the "alternate proof of Theorem I.B.13(b)" worked above.

I.B.20. REMARK (What about the chain rule?). Assume

$$f\colon U \to \mathbb{C}\,, \quad g\colon V \to U$$

are continuously $\mathbb{R}$-differentiable.[7] By writing $\partial/\partial z, \partial/\partial \bar{z}$ in terms of $\partial/\partial x$ and $\partial/\partial y$ and using the multivariable chain rule, one finds that

$$\frac{\partial}{\partial z}(f\circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\frac{\partial \bar{g}}{\partial z}(z) \tag{I.B.21}$$

and

$$\frac{\partial}{\partial \bar{z}}(f\circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial \bar{z}}(z) + \frac{\partial f}{\partial \bar{z}}(g(z))\frac{\partial \bar{g}}{\partial \bar{z}}(z). \tag{I.B.22}$$

A corollary is that if f or g is holomorphic, then

$$\frac{\partial}{\partial z}(f\circ g)(z) = \frac{\partial f}{\partial z}(g(z))\frac{\partial g}{\partial z}(z).$$

We may recover the usual partials from our operators via

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \qquad \frac{\partial}{\partial y} = \mathbf{i}\frac{\partial}{\partial z} - \mathbf{i}\frac{\partial}{\partial \bar{z}}. \tag{I.B.23}$$

This leads to the following

I.B.24. PROPOSITION. *If U is a connected open set, f is holomorphic, and $f' \equiv 0$ (identically zero), then f is constant.*[8]

SKETCH OF PROOF. First we note that

- f holomorphic $\implies \frac{\partial f}{\partial \bar{z}} \equiv 0$ (by (I.B.19))
- $f' \equiv 0 \implies \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} + \cancelto{0}{\frac{\partial f}{\partial \bar{z}}} = f' \equiv 0$ (again using (I.B.19))
- similarly, $\frac{\partial f}{\partial y} \equiv 0$.

Now any 2 points of U are connected by a chain of horizontal and vertical segments. Since the partials are zero in the horizontal and vertical directions, f doesn't change along any of these paths. $\square$

I.B.25. EXAMPLE. $f(z) = e^z = e^x \cos y + \mathbf{i}e^x \sin y$ is holomorphic: $\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = e^x \sin y = -\frac{\partial u}{\partial y}$.

[7] Here U and V are open subsets of $\mathbb{C}$.

[8] Once again, f' is $\frac{df}{dz}$, and these only have meaning for a holomorphic function. If f is not known to be holomorphic (e.g. if one is trying to prove f is holomorphic), only $\frac{\partial f}{\partial z}$ has meaning and one should not write f' or $\frac{df}{dz}$.

I.B.26. EXAMPLE. $f(z) = \bar{z} = x - \mathbf{i}y$ (so $u = x, v = -y$) is not holomorphic: $\frac{\partial u}{\partial x} = 1 \neq -1 = \frac{\partial v}{\partial y}$. Moreover, $\frac{\partial}{\partial \bar{z}}\bar{z} = 1$. (Other non-holomorphic functions include $|z|$, $|z|^2$.)

I.B.27. EXAMPLE. In contrast,

$$\frac{\partial}{\partial \bar{z}} z = \frac{1}{2}\left\{\frac{\partial z}{\partial x} + \mathbf{i}\frac{\partial z}{\partial y}\right\} = \frac{1}{2}\{1 + \mathbf{i}(\mathbf{i})\} = 0 \implies z \text{ is holomorphic.}$$

Since $\partial/\partial\bar{z}$ obeys the usual linearity,[9] product, and quotient rules, $\partial/\partial\bar{z}$ of every polynomial or (more generally) rational function of z is *zero*, making such functions holomorphic (where defined).

Further, the product rule and $\frac{\partial z}{\partial z} = 1$ together with induction yield

$$(z^n)' = \frac{\partial}{\partial z} z^n = nz^{n-1}.$$

§ Rational functions.

In this section we investigate some interesting properties of rational and polynomial functions, particularly with regard to their zeroes[10] and poles. We shall assume the *Fundamental Theorem of Algebra*, which will be proved later. Recall that this says that every polynomial of positive degree with complex coefficients has a complex root. Arguing inductively, this means that a polynomial of degree n breaks into n linear factors, and thus has n roots counted with multiplicity.

Let $P(z), Q(z)$ be polynomials with no common factors, and put

$$R(z) := \frac{P(z)}{Q(z)} \qquad \text{a rational function.}$$

We may extend this to a map $R\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting

$$\text{(I.B.28)} \qquad \begin{cases} R(\text{roots of } Q) := \infty \\ R(\infty) := \lim_{w\to 0} R\left(\frac{1}{w}\right). \end{cases}$$

[9] This means $\frac{\partial}{\partial \bar{z}}(\alpha f + \beta g) = \alpha\frac{\partial f}{\partial \bar{z}} + \beta\frac{\partial g}{\partial \bar{z}}$.

[10] For polynomial functions, I use the term "roots" interchangeably with "zeroes".

Define

- $\text{ord}_{z_0}(R)$:= the unique integer k which makes $(z-z_0)^{-k}R(z)$ finite and nonzero at $z = z_0$,
- $\text{ord}_\infty(R)$:= the unique integer k which makes $w^{-k}R\left(\frac{1}{w}\right)$ finite and nonzero at $w = 0$, and
- $\deg(R) := \max\{\deg P, \deg Q\}$.[11]

One easily checks that

$$\begin{aligned}\deg(R) &= \sum_{\substack{z\in\hat{\mathbb{C}}\\ \text{ord}_z(R)>0}} \text{ord}_z(R)\\ &= -\sum_{\substack{z\in\hat{\mathbb{C}}\\ \text{ord}_z(R)<0}} \text{ord}_z(R)\\ &= \begin{cases}\text{\# of solutions (with multiplicity)}\\ \qquad\text{of } R(z)=\alpha, \text{ for any } \alpha\in\mathbb{C}\end{cases}\end{aligned}$$

(The first sum counts zeroes, the second poles.) For example, if $\deg P < \deg Q$, then R has a zero of order $\deg Q - \deg P$ at ∞, and combined with the $\deg P$ zeroes of R coming from its numerator P, we have $\deg Q$ zeroes for R (and so the first sum is $\deg Q$). The sign in second expression is because R has negative order at its poles, but the reasoning is the same (replacing R by its reciprocal). For the last equality, notice that $R(z)-\alpha$ has the same poles and thus the same degree as $R(z)$.

Let $\{z_j\}$ be the finite poles of $R(z)$.

I.B.29. THEOREM (Partial fractions decomposition). *There exist polynomials P_j (of degree $-\text{ord}_{z_j}(R)$) and P_∞ (of degree $-\text{ord}_\infty(R)$) such that*

$$R(z) = \textstyle\sum_j P_j\left(\frac{1}{z-z_j}\right) + P_\infty(z).$$

(*One has the term $P_\infty(z)$ if and only if $\text{ord}_\infty(R) \le 0$.*)

---

[11] This is called the "order" of R in [**Ahlfors**], but "degree" is more consistent with modern usage. Note that $\deg P$ and $\deg Q$ are just the degrees of P and Q as polynomials.

PROOF. Since z_j is a pole of $R(z)$, the rational function $R\left(z_j + \frac{1}{w}\right)$ of w has a pole at $w = \infty$. Hence the degree of its numerator exceeds the degree of its denominator. Long division yields

$$R\left(z_j + \frac{1}{w}\right) = P_j(w) + R_j(w),$$

where P_j is a polynomial in w and

$$\deg(\mathrm{num}(R_j)) \leq \deg(\mathrm{den}(R_j)).$$

Substituting $w = \frac{1}{z-z_j}$ ($\Longrightarrow\ z_j + \frac{1}{w} = z$), we see

$$R(z) - P_j\left(\tfrac{1}{z-z_j}\right) = R_j\left(\tfrac{1}{z-z_j}\right)$$

is *finite* at z_j. Moreover, in passing from $R(z)$ to $R(z) - P_j(\frac{1}{z-z_j})$, no other poles have been introduced (or changed in order). Only the pole at z_j has disappeared. Continue this process for the remaining z_j's, so that finally

$$R(z) - \textstyle\sum_j P_j(\frac{1}{z-z_j})$$

has no poles in $\mathbb{C}$, and so must be a polynomial $P_\infty(z)$. $\square$

To conclude, we take a look at polynomials

$$P(z) = \alpha_n \prod_{j=1}^{n}(z - z_j) = \sum_{k=0}^{n} \alpha_k z^k.$$

Writing

$$P'(z) = \beta_{n-1} \prod_{\ell=1}^{n-1}(z - w_\ell) = \sum_{k=1}^{n} k\alpha_k z^{k-1} = \sum_{k=1}^{n} \beta_{k-1} z^{k-1},$$

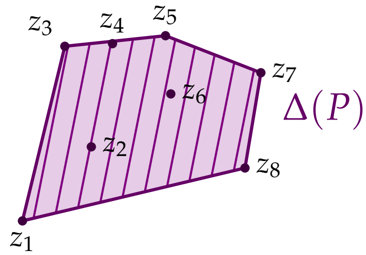
we notice that the *average*[12] of the zeroes in $\mathbb{C}$

$$\frac{\sum w_\ell}{n-1} = \frac{-\beta_{n-2}}{(n-1)\beta_{n-1}} = \frac{-(n-1)\alpha_{n-1}}{(n-1)n\,\alpha_n} = \frac{-\alpha_{n-1}}{n\,\alpha_n} = \frac{\sum z_j}{n}$$

doesn't change with differentiation of P.

[12]This is weighted by multiplicity by construction, so is a "center of mass" of the zeroes if you will.

This suggests looking at the **convex hull** of the zeroes, by which we mean the *smallest convex polygon* $\Delta(P)$ *containing all the* z_j*'s*.



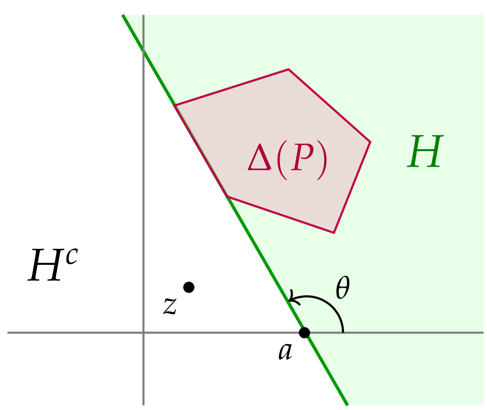
I.B.30. THEOREM (Lucas, 1874).

$$\Delta(P') \subseteq \Delta(P).$$

PROOF. $\Delta(P)$ is an intersection of closed half-planes; so it suffices to show each w_ℓ belongs to any closed half-plane H containing all the $\{z_j\}$. Put differently, given any closed half-plane H containing the $\{z_j\}$, we must show that P' *has no zeroes in the (open) complement* H^c. Since P has no zeroes there, we can check the assertion for a constant times

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - z_j}.$$

If $z \notin H$, then referring to the picture



we have $\mathrm{Im}(\frac{z-a}{e^{\mathrm{i}\theta}}) > 0$, while $z_j \in H \implies \mathrm{Im}(\frac{z_j-a}{e^{\mathrm{i}\theta}}) \leq 0$. (Imagine translating the figure a units to the left, then rotating about the origin *clockwise* by θ; this moves H^c exactly to the lower half-plane $y < 0$.)

Putting these together gives

$$0 < \mathrm{Im}\left(\frac{z-a}{e^{\mathbf{i}\theta}}\right) = \mathrm{Im}\left(\frac{z_j-a}{e^{\mathbf{i}\theta}}\right) + \mathrm{Im}\left(\frac{z-z_j}{e^{\mathbf{i}\theta}}\right)$$
$$\leq \mathrm{Im}\left(\frac{z-z_j}{e^{\mathbf{i}\theta}}\right).$$

Thus $\mathrm{Im}\left(\frac{e^{\mathbf{i}\theta}}{z-z_j}\right) < 0$, as the imaginary part of the reciprocal has the opposite sign (think $re^{\mathbf{i}\phi} \mapsto \frac{1}{r}\overline{e^{\mathbf{i}\phi}}$). Therefore for $z \notin H$,

$$\mathrm{Im}\left(\frac{e^{\mathbf{i}\theta}P'(z)}{P(z)}\right) = \sum_{j=1}^{n} \mathrm{Im}\left(\frac{e^{\mathbf{i}\theta}}{z-z_j}\right) < 0,$$

which implies $P'(z) \neq 0$, completing the proof. □

Exercises

(1) Let $S := \{x + \mathbf{i}y \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\} \subset \mathbb{C}$, and let $f(z) := e^{-\pi \mathbf{i} z}$. Find (a) the image $f(S)$ and (b) the preimage $f^{-1}(1)$.

(2) Prove that $\log(z) := \log|z| + \mathbf{i}\arg(z)$ is holomorphic on the open set $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ (use the Cauchy-Riemann equations).

(3) Write out a the detailed argument for (I.B.18), and complete the argument for (I.B.19). (To get all the implications, you will need to invoke Theorem I.B.13.)

(4) Prove (I.B.21) using the argument suggested in Remark I.B.20.

(5) Let f be a holomorphic function (on a connected open set) with constant absolute value. Show that f is constant. [Hint: apply $\frac{\partial}{\partial z}$ to $|f|^2$.]