I.C. Topology of the complex plane

§ Topological spaces.

We begin with some generalities.

I.C.1. DEFINITION. A **topological space** is a pair (X, Σ), where X is a set (of "points"), and Σ is a collection of subsets of X including:

(a) X itself and the empty set $\emptyset$;
(b) finite intersections $\bigcap_{j \in J} U_j$ if each $U_j \in \Sigma$; and
(c) arbitrary unions $\bigcup_{j \in J} U_j$ if each $U_j \in \Sigma$.

Σ is called a **topology** on X; members of Σ are called **open sets** (and their complements are **closed sets**).

I.C.2. REMARKS.

(i) The notation for the complement of $U \in \Sigma$ is U^c or $X \setminus U$.

(ii) We will frequently write "X" instead of "(X, Σ)" for topological spaces.

Some more terminology:

- X is **Hausdorff** $\iff$ for any distinct $x, y \in X$, there exist disjoint open sets $U, V \in \Sigma$ with $U \ni x$ and $V \ni y$.
- A **basis** (or **base**) for the topology is a "generating set" $\mathcal{B} \subseteq \Sigma$; i.e. every $U \in \Sigma$ is a union of elements of $\mathcal{B}$.
- Let $S \subseteq X$ be a subset (not necessarily belonging to Σ). We denote

$$(S \supseteq)\mathring{S} := \{x \in X \mid \exists U \in \Sigma \text{ such that } x \in U \subseteq S\} \quad (\textbf{interior} \text{ of } S)$$

$$\partial S := \{x \in X \mid \forall U \in \Sigma \text{ such that } x \in U, \; U \cap S \neq \emptyset \neq U \cap S^c\} \quad (\textbf{boundary} \text{ of } S)$$

$$\operatorname{acc}(S) := \{x \in X \mid \forall U \in \Sigma \text{ s.t. } x \in U, \; (U \setminus \{x\}) \cap S \neq \emptyset\} \quad (\textbf{accumulation points} \text{ of } S)$$

$$\overline{S} := \{x \in X \mid \forall U \in \Sigma \text{ s.t. } x \in U, \; U \cap S \neq \emptyset\} \quad (\textbf{closure} \text{ of } S)$$
$$= \mathring{S} \sqcup \partial S \supseteq \operatorname{acc}(S).$$

Note that $\operatorname{acc}(S)$ $(\subseteq \overline{S})$ is also called the set of **limit points** of S.

- Subspace topology: $\Sigma_S := \{U \cap S \mid U \in \Sigma\}$.

- S is **connected** $\iff$ S cannot be written as the disjoint union of nonempty $U, V \in \Sigma_S$.
- S is **compact** $\iff$ every open cover has a finite subcover: i.e. $\forall \{U_j\}_{j\in J} \subseteq \Sigma$ s.t. $S \subseteq \bigcup_{j\in J} U_j$, $\exists j_1, \ldots, j_n \in J$ s.t. $S \subseteq \bigcup_{i=1}^n U_{j_i}$.

I.C.3. REMARK. Any $S \subseteq X$ has a unique decomposition into connected components (= maximal connected subsets). As these are both closed and open in S, they are sometimes called **clopen**.

Distance. If $d \colon X \times X \to \mathbb{R}$ is

- symmetric and nonnegative, with
- $d(x, y) = 0 \iff x = y$, and
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality),

then we shall call it a **distance function**. Given a distance function, it makes sense to take as a basis for Σ the "open disks"

$$D(x_0, r) := \{x \in X \mid d(x, x_0) < r\}.$$

When Σ is constructed in this way, the triple (X, Σ, d) is called a **metric space**.

I.C.4. REMARK. I will also use the notation $\overline{D}(x_0, r) := \{x \in X \mid d(x, x_0) \leq r\}$ and $D^*(x_0, r) := \{x \in X \mid 0 < d(x, x_0) < r\}$.

I.C.5. EXAMPLES. Let $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half-plane and $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disk. Just to demonstrate that there is more to life than the usual Euclidean distance function, here without explanation are several metrics arising in 1-variable complex analysis.

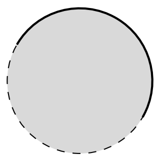
X	name of metric	$d(z_1, z_2)$
$\mathbb{C}$	Euclidean	$\|z_1 - z_2\|$
$\hat{\mathbb{C}}$	stereographic	$\frac{2\|z_1-z_2\|}{\sqrt{(1+\|z_1\|^2)(1+\|z_2\|^2)}}$
$\mathfrak{H}$	Poincaré	$\log\left(\frac{\|z_1-\overline{z_2}\|+\|z_1-z_2\|}{\|z_1-\overline{z_2}\|-\|z_1-z_2\|}\right)$
Δ	Poincaré	$\tanh^{-1}\left(\frac{\|z_1-z_2\|}{\|1-z_1\overline{z_2}\|}\right)$

We'll investigate the Poincaré metrics later in the course.

§ The complex plane.

For now we'll work on $X = \mathbb{C}$ with the Euclidean metric $d(x, y) = |x - y|$. In particular, open sets contain discs around each of their points.

I.C.6. EXAMPLES (Subsets of $\mathbb{C}$).

(1) S: 

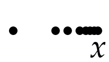
$\partial S =$ *entire* circle
$\mathring{S} =$ shaded area
$\mathrm{acc}(S) = \overline{S} = \mathring{S} \sqcup \partial S$

(2) S: 

$S = \overline{S}$ (S is closed)
$\mathrm{acc}(S) = \emptyset$

(3) $S = \mathfrak{H}$:

(upper half plane)
$S = \mathring{S}$(open set)

(4) S:  $\mathrm{acc}(S) = \{x\}$, whether or not $x \in S$

I.C.7. REMARK. (i) The definition of $x \in \mathrm{acc}(S)$ says that for any $\epsilon > 0$, $D^*(x, \epsilon) \cap S$ is nonempty. In effect, this means that $D^*(x, \epsilon) \cap S$ contains infinitely many points. For $X = \mathbb{C}$, $\mathring{S} \subseteq \mathrm{acc}(S) \subseteq \overline{S}$ but we need not have $\mathrm{acc}(S) \subseteq S$ or $S \subseteq \mathrm{acc}(S)$ (cf. Examples (2), (4) above).

(ii) Given $S \subseteq \mathbb{C}$, the open subsets of S are the $\{S \cap U\}$ for $U \in \Sigma$ (i.e. U open in $\mathbb{C}$); the closed subsets are their complements (in S). But $S \cap U$ (resp. $S \cap U^c$) may not be open (resp. closed) *in* $\mathbb{C}$. In this sense, openness or closedness are *relative* properties. (From the form of the definition of *compactness*, it is clear that this is an *absolute* property.)

I.C.8. EXAMPLES (More subsets of $\mathbb{C}$).

(a) $S = \mathbb{R}$: $U = \{z \mid \mathrm{Re}(z) \in (1, 2)\}$ is open in $\mathbb{C}$, so the interval $I = U \cap S = (1, 2)$ is open in S. However, I is not open in $\mathbb{C}$.

(b) $S = \mathfrak{H}$: the set $V = \{z \mid \mathrm{Im}(z) > 0\}$ is closed in S (why?), but V is *not* closed as a subset of $\mathbb{C}$.

§ **Limits**.

Let $f\colon S \to \mathbb{C}$, $\alpha \in \mathrm{acc}(S)$ and $\beta \in \mathbb{C}$ be given.

I.C.9. DEFINITION. We say that

$$\lim_{S\ni z\to\alpha} f(z) = \beta$$

if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(S \cap D(\alpha,\delta)) \subseteq D(\beta,\epsilon).$$

Writing $S\ni z$ is a bit fussy, but if f is the restriction to S of a function defined on a larger set, taking the limit "in S" can make a difference as to whether the limit is defined.

Now suppose $\alpha \in S$ so that $f(\alpha)$ is defined.

I.C.10. DEFINITION. f is **continuous** at α if

$$\lim_{S\ni z\to\alpha} f(z) = f(\alpha).$$

A sequence $\{z_n\}$ in $\mathbb{C}$ amounts to a (continuous) function

$$f\colon \underbrace{\{\tfrac{1}{n} \mid n \in \mathbb{Z}_+\}}_{=:S} \to \mathbb{C},$$

and the limit of the sequence (if it exists) is given by

$$\lim_{n\to\infty} z_n = w \overset{\text{def}}{\iff} \lim_{S\ni z\to 0} f(z) = w.$$

The **accumulation points** (or **limit points**) of the sequence are all the *limits of subsequences* $\{z_{n_k}\}$. (Note that the accumulation points $\mathrm{acc}(\{z_n\})$ of the *set* $\{z_n\}_{n\in\mathbb{Z}_+}$ may be different — e.g., for constant sequences, empty!)

I.C.11. DEFINITION. A sequence is **Cauchy** if for every $\epsilon > 0$ there is an $N \in \mathbb{Z}_+$ such that

$$n, m \geq N \Rightarrow |z_n - z_m| < \epsilon.$$

Every Cauchy sequence in $\mathbb{C}$ converges due to the *completeness property* of $\mathbb{C}$.

§ Remarks on connectedness.

Connectedness was defined previously.

Real numbers. $I \subset \mathbb{R}$ is connected if and only if I is an interval. Consequently, any bounded-above [resp. below], nonempty subset $S \subseteq \mathbb{R}$ has a **least upper bound** [resp. greatest lower bound], since the set of all upper bounds for S is easily shown to be connected and closed. More precisely: $\mathbf{lub}(S) =: s$ is the unique real number satisfying

(i) $s \geq$ every element of S; and

(ii) if $t \geq$ every element of S, then $t \geq s$ as well.

(See **[Ahlfors]** for details.)

Complex numbers. For $S \subset \mathbb{C}$, a **path** in S is a continuous function $\gamma\colon [0,1] \to S$, and the important thing to remember is:

I.C.12. PROPOSITION. *Assume $S \subseteq \mathbb{C}$ is open. Then S is connected if and only if S is **pathwise connected** (i.e. for every pair $\alpha, \beta \in S$, there exists a path γ with $\gamma(0) = \alpha, \gamma(1) = \beta$).*

SKETCH. ($\Longrightarrow$) $S_0 \subseteq S$ maximal path connected $\Longrightarrow$ S_0 open & closed (use open balls) $\Longrightarrow$ $S_0 \subseteq S$ maximal connected (so $S_0 = S$).

($\Longleftarrow$) Suppose S is path connected. If S is not connected, we can write it as a disjoint union $U \sqcup V$ of open sets. Pick $\alpha \in U$ and $\beta \in V$, and let γ be a path in S with $\gamma(0) = \alpha$ and $\gamma(1) = \beta$. Then $[0,1] = \gamma^{-1}(U) \sqcup \gamma^{-1}(V)$ is also not connected, a contradiction. □

A connected open set in $\mathbb{C}$ is called a **region**, and it is on such sets that we will generally want to study holomorphic functions.

§ Boundedness.

I.C.13. DEFINITION. $S \subseteq \mathbb{C}$ is bounded if there is a positive real number C such that $|s| \leq C$ ($\forall s \in S$).

I.C.14. THEOREM (Bolzano–Weierstrass). *If S is a bounded infinite subset of $\mathbb{C}$, then* $\mathrm{acc}(S)$ *is nonempty* ("*S has limit points*").

PROOF. Let $\{r_n\} \subset \mathbb{R}$ be a bounded sequence. We can extract either a nonincreasing or nondecreasing subsequence: if the subset of $\{r_n\}$ consisting of "elements $\geq$ all subsequent terms" is infinite, it gives a nonincreasing sequence. Otherwise, one can construct a nondecreasing one. Now take glb or lub of that subsequence; this exists and must be its limit s.

Next let $\{s_n\} \subset S$ be an arbitrary sequence of *distinct* elements (which exists as $|S| = \infty$). We may extract a subsequence with convergent real part, then a sub-subsequence with convergent imaginary part (by the result for $\{r_n\}$ above). The latter $\{s_{n_k}\}$ must then not only converge to some $s \in \mathbb{C}$, but have $s \in \mathrm{acc}(\{s_{n_k}\}) \subseteq \mathrm{acc}(S)$ since all the s_{n_k} are distinct. $\square$

I.C.15. THEOREM. *The following are equivalent for a subset $K \subseteq \mathbb{C}$:*

(i) K is compact;

(ii) Any sequence $\{z_n\} \subseteq K$ has a point of accumulation in K (equivalently: a convergent subsequence with limit in K); and

(iii) K is closed (i.e. $K = \overline{K}$) *and bounded.*

PROOF. (ii) $\Rightarrow$ (i): Exercise.

(iii) $\Rightarrow$ (ii): See proof of Bolzano-Weierstrass.

(i) $\Rightarrow$ (iii): *Boundedness:* take $U_n = \{|z| < n\}$; then $\bigcup U_n = \mathbb{C} \supset K$ ($\Rightarrow$ $\{U_n\}$ gives open cover). K compact $\Rightarrow$ $K \subseteq U_N$ for some N. *Closedness:* assume K not closed. Take $p \in \partial K \cap K^c$ and consider $U_n = (\overline{D}(p, \frac{1}{n}))^c$. Then $\bigcup U_n = \mathbb{C} \backslash \{p\} \supseteq K$, and K compact $\Rightarrow$ $K \subseteq U_N$. But since $p \in \partial K$, U_N^c must contain a point of K, a contradiction. $\square$

I.C.16. REMARK. Any accumulation point of $\{z_n\} \subseteq K$ compact is in K, since K is closed.

I.C.17. THEOREM. *Let $K \subset \mathbb{C}$ be compact, with nonempty closed nested subsets $(K \supseteq) S_1 \supseteq S_2 \supseteq \ldots$. Then $\bigcap S_n \neq \emptyset$.*

PROOF. The S_n are compact (they are closed, and inherit boundedness from K). For each n, choose $z_n \in S_n$; K contains an accumulation point of $\{z_n\}$, say α. This α is an accumulation point of tails $\{z_m\}_{m \geq n} \subseteq S_n$, hence $\alpha \in \overline{S_n} = S_n$ ($\forall n$). Hence $\bigcap S_n$ contains α. $\square$

§ Continuous functions on compact sets.

A first input of topology into complex function theory is the

I.C.18. THEOREM. *Let $S \subseteq \mathbb{C}$ be compact, and let $f\colon S \to \mathbb{C}$ be continuous. Then*

(i) $|f|$ attains a maximum on S,
(ii) f is uniformly continuous, and
(iii) $f(S)$ is compact.

Before commencing the proof, which will also work for $\mathrm{Re}(f)$, $\mathrm{Im}(f)$, etc., we need the

I.C.19. DEFINITION. f is **uniformly continuous** if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$z, w \in S \text{ and } |z - w| < \delta \quad \Longrightarrow \quad |f(z) - f(w)| < \epsilon.$$

PROOF. (iii) Let $\{w_n\} \subseteq f(S)$ be any sequence; then $w_n = f(z_n)$ for some sequence $\{z_n\} \subseteq S$. Since S is compact, Theorem I.C.15 and Remark I.C.16 provide a convergent subsequence $z_{n_k} \to v \in S$. Since f is continuous, $w_{n_k} = f(z_{n_k}) \to f(v) \in f(S)$ yields a point of accumulation for w_n in $f(S)$. Again invoking Theorem I.C.15, this implies $f(S)$ is compact.

(i) Now (iii) holds, by the same proof, for continuous real-valued functions on S, in particular the composition $S \xrightarrow{f} \mathbb{C} \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$. So $|f(S)|$ is compact, i.e. closed and bounded. Since it is bounded, $b := \mathrm{lub}(|f(S)|)$ must exist. Since it is closed, we have $b \in |f(S)|$, and there exists $v \in S$ with $|f(v)| = b$. Since b is an upper bound, $|f(v)| \geq |f(z)|\ \forall z \in S$.

(ii) Assume that f is *not* uniformly continuous. Then there exists some $\epsilon > 0$ for which no $\delta > 0$ has the desired property: if (for each $n \in \mathbb{Z}_{>0}$) we take $\delta_n = \frac{1}{n}$, then there exist z_n and w_n in S with $|z_n - w_n| < \delta_n$ but $|f(z_n) - f(w_n)| \geq \epsilon$. As S is compact, there exists $\{n_k\} \subset \mathbb{N}$ such that $z_{n_k} \to v \in S$ and $w_{n_k} \to u \in S$, whence

$$|v - u| \leq |v - z_{n_k}| + |z_{n_k} - w_{n_k}| + |w_{n_k} - u| \to 0.$$

This forces $v = u \implies f(v) = f(u) \implies$

$$|f(z_{n_k}) - f(w_{n_k})| \leq |f(z_{n_k}) - f(v)| + |f(u) - f(w_{n_k})|,$$

in which the right-hand side goes to 0 by continuity of f. But this contradicts $|f(z_{n_k}) - f(w_{n_k})| \geq \epsilon$. □

Here is a quick application of this Theorem: let $A, B \subseteq \mathbb{C}$ be subsets, and define

$$d(A,B) := \mathrm{glb}\{|z-w| \mid z \in A, w \in B\}.$$

I.C.20. COROLLARY. *(i) S closed and $v \in \mathbb{C} \setminus S \implies$ there exists a point $w_0 \in S$ with $d(S,v) = |w_0 - v|$.*

(ii) S closed and K compact $\implies$ there exist $z_0 \in K$ and $w_0 \in S$ such that $d(K,S) = |z_0 - w_0|$.

§ APPENDIX: An interesting non-Hausdorff space.

A more general definition of the limit of a sequence, for arbitrary topological spaces (X, Σ), is

$$\text{(I.C.21)} \qquad \lim_{n\to\infty} x_n = x \iff \begin{array}{c}\text{each } U \in \Sigma \text{ containing } x \text{ contains} \\ \text{all but finitely many of the } x_n.\end{array}$$

It turns out that uniqueness of the limit is implied by Hausdorffness of (X, Σ). (See the Exercises.) Hence, we can show that a topological space is non-Hausdorff by exhibiting a "non-unique limit point" — i.e. a sequence with two distinct limits in the sense of (I.C.21).

I.C.22. EXAMPLE. Writing $\Delta := D(0,1)$, $\Delta^* := D^*(0,1)$, and $\ell(s) := \frac{\log(s)}{2\pi\mathbf{i}}$, define a lattice $\Gamma_s \subseteq \mathbb{C}^2$ for each $s \in \Delta$ by the formulas

- $\Gamma_s := \mathbb{Z}\left\langle \begin{pmatrix} \mathbf{i} \\ \mathbf{i}\ell(s) \end{pmatrix}, \begin{pmatrix} 0 \\ \mathbf{i} \end{pmatrix}, \begin{pmatrix} 1 \\ \ell(s) \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$, for $s \in \Delta^*$; and
- $\Gamma_0 := \mathbb{Z}\left\langle \begin{pmatrix} 0 \\ \mathbf{i} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$, for $s = 0$.

Then

$$\bigcup_{s\in\Delta} \Gamma_s \times \{s\} \ =:\ \Gamma \subseteq \mathbb{C}^2 \times \Delta$$

(with the Euclidean topology as a subset of $\mathbb{C}^3$) is reasonably nice (e.g. union of smooth submanifolds), but

$$\hat{X} := \frac{\mathbb{C}^2 \times \Delta}{\Gamma}$$

(equipped with the quotient topology) is *non-Hausdorff*.

SKETCH. Fix $a \in \mathbb{Z}_+$ and $b \in \mathfrak{H}$; then $s_n := e^{2\pi\mathbf{i}(b+n\mathbf{i})/a} \in \Delta^*$ has $\lim_{n\to\infty} s_n = 0$.

Let $v_n := a \begin{pmatrix} 1 \\ \ell(s_n) \end{pmatrix} - n \begin{pmatrix} 0 \\ \mathbf{i} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$.

Then $(v_n, s_n) \in \Gamma_{s_n} \times \{s_n\} \subset \Gamma \subset \mathbb{C}^2 \times \Delta$ has $\lim_{n\to\infty}(v_n, s_n) = \left(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), 0\right)$. On the other hand, $\lim_{n\to\infty}\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), s_n\right) = \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0\right)$.

Under the quotient topology on $\hat{X}$, it remains the case that

$$\lim_{n\to\infty}(v_n, s_n) = \left(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), 0\right) \quad \text{and} \quad \lim_{n\to\infty}\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0\right) = \left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0\right).$$

On the other hand, (v_n, s_n) and $\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), s_n\right)$ are literally the same sequence in $\hat{X}$, since $v_n \in \Gamma_{s_n}$; while $\left(\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right), 0\right)$ and $\left(\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0\right)$ are distinct points of $\hat{X}$ because $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ does not belong to Γ_0. $\square$

I.C.23. REMARK. Easier examples of non-Hausdorff spaces are available, if one is willing to leave the realm of complex geometry. (For instance, the Zariski topology on nearly any algebraic variety is non-Hausdorff.)

Exercises

(1) Let X be a topological space, $S \subseteq X$ a subset.
(i) Prove that S is open $\iff$ $S = \mathring{S}$. (If $X = \mathbb{C}$, deduce that S is open $\iff$ it contains an open disk about every point.)
(ii) Prove that S is closed $\iff$ $S \subseteq \mathrm{acc}(S)$ $\iff$ $S = \overline{S}$.

(2) State precisely and prove: "sequential limits are unique in a Hausdorff topological space".

(3) Show that the union of two regions is a region if and only if they have a common point.

(4) Given $K \subseteq \mathbb{C}$ with the property that any sequence in K has a point of accumulation in K, prove directly that K is compact.

(5) (a) On what sets (in $\mathbb{C}$) does $f_n(z) := \frac{z^n-1}{z^n+1}$ converge uniformly?

(b) On what sets (in $\mathbb{R}$) is $|x|^{\frac{1}{2}} \sin x$ uniformly continuous?