I.D. Power series

§ Convergence of function series.

Let $S \subseteq \mathbb{C}$ be a subset, and denote

$$\mathcal{F}(S) := \mathbb{C}\text{-valued functions on } S,$$
$$\mathcal{C}^0(S) := \text{continuous } \mathbb{C}\text{-valued functions on } S.$$

I.D.1. DEFINITION (Convergence of series).

(i) **Convergence** of series: for $a_k \in \mathbb{C}$, we write $\sum_{k=0}^{\infty} a_k = L$ if the sequence $\{\sum_{k=0}^{n} a_k\}$ has limit L.

(ii) **Pointwise convergence** of function series: for $f_k, F \in \mathcal{F}(S)$, we write $\sum_{k=0}^{\infty} f_k = F$ if for every $z \in S$, the sequence $\{\sum_{k=0}^{n} f_k(z)\}$ has limit $F(z)$.

(iii) **Absolute convergence** of $\sum a_k$ (resp. $\sum f_k$) means convergence of $\sum |a_k|$ (resp. $\sum |f_k|$).

I.D.2. PROPOSITION. *Absolute convergence* $\implies$ *convergence.*

PROOF. Given $\epsilon > 0$. Since $z_n := \sum_{k=0}^{n} |a_k| \to z$, we can choose N sufficiently large that $n > m \geq N$ implies

$$\epsilon > |z - z_m| = z - z_m = \sum_{k=m+1}^{\infty} |a_k| \geq \sum_{k=m+1}^{n} |a_k|.$$

Set $s_n := \sum_{k=0}^{n} a_k$; then $n > m \geq N$ implies

$$|s_n - s_m| = |a_{m+1} + \cdots + a_n| \leq \sum_{k=m+1}^{n} |a_k| < \epsilon.$$

So $\{s_n\}$ is Cauchy, hence convergent (by completeness of $\mathbb{C}$). $\square$

I.D.3. DEFINITION (Uniform convergence). For $f \in \mathcal{F}(S)$ bounded, the **sup norm**[13] is given by

$$\|f\|_S := \text{lub}_{z \in S}\{|f(z)| \mid z \in S\}.$$

[13]Also called the L^{∞} norm, this may be regarded as a distance function on bounded functions on S.

We say that a sequence $\{f_n\} \subseteq \mathcal{F}(S)$ **converges uniformly** to $f \in \mathcal{F}(S)$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies \|f_n - f\|_S < \epsilon.$$

Uniform convergence implies (pointwise) convergence since

$$\|f_n - f\|_S \geq |f_n(z_0) - f(z_0)| \qquad (\forall z_0 \in S).$$

I.D.4. THEOREM. *(i)* $\{f_n\}$ *Cauchy in sup norm* $\implies$ $\{f_n\}$ *converges uniformly (i.e. in sup norm)*

(ii) If $\{f_n\} \subset \mathcal{C}^0(S)$ *converges uniformly to* $f \in \mathcal{F}(S)$, *then* $f \in \mathcal{C}^0(S)$.

PROOF. (i) $\{f_n(z)\}$ Cauchy $\forall z \in S \implies$ we can put $f(z) := \lim_{n\to\infty} f_n(z)$. Take $\epsilon > 0$ and N sufficiently large that $|f_n(z) - f_m(z)| < \frac{\epsilon}{2}$ (for all $m, n \geq N$ and $z \in S$). For *any particular* z_0, we can take m sufficiently large that also $|f(z_0) - f_m(z_0)| < \frac{\epsilon}{2}$, hence that $|f(z_0) - f_n(z_0)| < \epsilon$ ($\forall n \geq N$). But N doesn't depend on z_0.

(ii) Take $\epsilon > 0$ and $z_0 \in S$. There exist

- n sufficiently large that $\|f - f_n\|_S < \frac{\epsilon}{3}$
- $\delta > 0$ sufficiently small that $|f_n(z) - f_n(z_0)| < \frac{\epsilon}{3}$ for $|z - z_0| < \delta$.

Hence, $|f(z) - f(z_0)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ for $|z - z_0| < \delta$. $\square$

We say that $\sum f_n$ **converges uniformly** if the sequence of partial sums does.

I.D.5. THEOREM (Weierstrass M-test). *Suppose* $\{M_n\} \subset \mathbb{R}_{\geq 0}$, $\sum M_n$ *converges,* M_n *bounds* $|f_n|$ *on* S, *and* $f_n \in \mathcal{C}^0(S)$. *Then* $\sum f_n \in \mathcal{C}^0(S)$.

PROOF. Let $\epsilon > 0$. Then $n > m \geq N$ sufficiently large $\implies$

$$\epsilon > M_{m+1} + \cdots + M_n \geq \|f_{m+1}\|_S + \cdots + \|f_n\|_S \geq \|f_{m+1} + \cdots + f_n\|_S$$

$\implies$ $\sum_{k=0}^n f_n =: s_n$ is Cauchy in $\|\cdot\|_S$, hence converges uniformly. But $s_n \in \mathcal{C}^0(S) \implies \lim_{n\to\infty} s_n = \sum f_n$ is too by Thm. I.D.4(ii). $\square$

§ Convergent power series.

The M-test has as an immediate consequence:

I.D.6. COROLLARY. *Given* $\{a_n\} \subset \mathbb{C}$ *and* $s \in \mathbb{R}_{\geq 0}$ *such that* $\sum |a_n| s^n$ *converges,* $\sum a_n z^n$ *converges absolutely and uniformly, and defines a continuous function on* $|z| \leq s$.

Define the **radius of convergence** of $\sum a_n z^n$ by

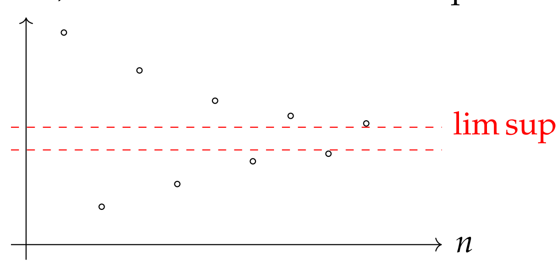
$$r := \text{lub}\{s \in \mathbb{R}_{\geq 0} \mid \sum |a_n| s^n \text{ converges}\}.$$

By the Corollary, $\sum |a_n||z^n|$ and $\sum a_n z^n$ are convergent for $|z| < r$, and divergent for $|z| > r$. We give two formulas for the radius of convergence below.

Recall that for a *bounded* sequence $\{t_n\} \subseteq \mathbb{R}$,

$$\limsup t_n := \lim_{n\to\infty} \text{lub}\{t_k \mid k \geq n\} = \text{lub}\{\text{accumulation pts. of } \{t_n\}\}.$$

Since the accumulation points constitute a nonempty and bounded set, the lim sup exists. (In the event that the sequence belongs to $\mathbb{R}_{\geq 0}$ and is unbounded, the lim sup is defined to be ∞.) Here is one possible situation, where there are two subsequential limits:



For the radius, we then have:

I.D.7. PROPOSITION. *(i)* [*root test*] $r = (\limsup |a_n|^{1/n})^{-1}$
(ii) [*ratio test*] $r = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ *if it exists.*

The proof of the ratio test is by a standard comparison to geometric series; instead we focus on the

PROOF OF THE ROOT TEST. Set $A_n := |a_n|^{1/n}$, $A := \limsup A_n \in [0,\infty) \cup \{\infty\}$. Notice that this always exists.

CASE 1 ($\{A_n\}$ unbounded $\iff A = \infty$). Clearly $(|a_n|s^n)^{1/n} = A_n s$ is unbounded for any $s \in \mathbb{R}_{>0}$. Therefore $|a_n|s^n$ is unbounded $\implies \sum |a_n|s^n$ doesn't converge ($\forall s \in \mathbb{R}_{>0}$) $\implies r = 0$.

CASE 2 ($\{A_n\}$ bounded $\iff A \in [0,\infty)$). Here $\{A_n\} \subseteq [0,B]$ for some $B \in \mathbb{R}_{\geq 0}$. If infinitely many $A_n \geq A + \epsilon$, one would have a subsequence $\{A_{n_k}\} \subset [A+\epsilon, B]$, with sub-subsequence converging to a number in $[A+\epsilon, B]$ (Bolzano-Weierstrass). This is impossible as $A = \text{lub}\{\text{limits of subsequences of } \{A_n\}\}$. Therefore, given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$n \geq N \implies A_n < A + \epsilon \ (\implies |a_n| < (A+\epsilon)^n).$$

So for some $C > 0$, $|a_n| < C(A+\epsilon)^n$ for *all* n; and taking $s < \frac{1}{A+\epsilon}$,

$$\sum |a_n|s^n \leq \sum C(A+\epsilon)^n s^n = \frac{C}{1-(A+\epsilon)s}. \tag{I.D.8}$$

Since $r = \text{lub}\{\text{those } s \text{ for which LHS(I.D.8) converges}\}$, $r \geq \frac{1}{A+\epsilon}$. As $\epsilon > 0$ was arbitrary, we find $r \geq \frac{1}{A}$ if $A \in (0,\infty)$ and $r = \infty$ if $A = 0$. In the latter case, we are done.

Now if $A \in (0,\infty)$, we must have infinitely many $A_n \geq A - \epsilon$ ($\implies |a_n| \geq (A-\epsilon)^n$). Thus $\sum |a_n|s^n$ *cannot* converge for $s \geq \frac{1}{A-\epsilon}$, and so $r \leq \frac{1}{A-\epsilon}$. Again, since $\epsilon > 0$ was arbitrary, $r \leq \frac{1}{A}$. □

I.D.9. COROLLARY (to the root test). *Let $b > \frac{1}{r}$. Then*

$$\limsup \frac{|a_n|}{b^n} = \left(\limsup \frac{|a_n|^{1/n}}{b}\right)^n = \lim_{n\to\infty} \left(\frac{1/r}{b}\right)^n (= 0) < 1,$$

and so for $n \geq N$, $|a_n| \leq b^n$. Throw in a constant to get this to hold for $a_0, \ldots, a_{N-1}$: for appropriate $C > 0$,

$$|a_n| \leq b^n \cdot C \qquad (\forall n).$$

The existence of a bound of this form characterizes convergent power series.

§ Examples of power series.

(1) What does a *non*-convergent power series look like?

- $\sum nz^n$? No — that would ruin derivatives of power series!
- $\sum n^n z^n$? Yes — $A = \limsup |n^n|^{1/n} = \limsup n = \infty \implies r = 0$.
- $\sum e^n z^n$? $A = \limsup |e^n|^{1/n} = e \implies r = \frac{1}{e}$. Nope.
- $\sum n! z^n$? Use $(2n)! \geq (2n)(2n-1)\cdots n \geq n^n \implies ((2n)!)^{1/2n} \geq (n^n)^{1/2n} = \sqrt{n} \to \infty \implies r = 0$. So, yes, that's another example.
- $\sum \frac{n!}{n^n} z^n$??? It's convergent, but we need a more powerful tool:

(2) Stirling's formula (easy version): since the antiderivative of $\log(x)$ is $x\log(x) - x$,

$$\textstyle\sum_{k=2}^{n} \log k \geq \int_1^n \log(x)dx \geq \sum_{k=1}^{n-1} \log k$$

$$\implies \log(n!) \geq n\log(n) - n + 1 \geq \log((n-1)!)$$

$$\implies n! \geq n^n \cdot e^{-n} \cdot e \geq (n-1)! \tag{I.D.10}$$

Taking $u_n := \frac{n!}{n^n e^{-n}}$, we have $n! = u_n n^n e^{-n}$ and $(n-1)! = u_n n^{n-1} e^{-n}$, whence (I.D.10) leads easily to

$$(ne)^{1/n} \geq u_n^{1/n} \geq e^{1/n}. \tag{I.D.11}$$

Applying $n^{\frac{1}{n}} \to 1$ and the squeeze lemma to (I.D.11) yields

$$\lim_{n\to\infty} u_n^{1/n} = 1, \tag{I.D.12}$$

a weak form of Stirling. For $\sum a_n z^n = \sum \left(\frac{n!}{n^n}\right) z^n$, we can now compute

$$\frac{1}{r} = \limsup |a_n|^{1/n} = \lim_{n\to\infty} \left| \frac{u_n \cancel{n^n} e^{-n}}{\cancel{n^n}} \right|^{1/n} = \frac{1}{e} \implies r = e.$$

(3) Binomial and hypergeometric series:

(a) For any $\alpha \in \mathbb{C}$, define

$$(1+z)^\alpha := \sum_{n\geq 0} \binom{\alpha}{n} z^n, \tag{I.D.13}$$

where

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \tag{I.D.14}$$

(especially useful for $\alpha = \frac{1}{m}$). Where exactly does (I.D.13) define a function? According to the ratio test, it converges on the unit disk:

$$\left|\frac{\binom{\alpha}{n}}{\binom{\alpha}{n+1}}\right| = \left|\frac{\alpha\cdots(\alpha-n+1)}{\alpha\cdots(\alpha-n)}\right| \frac{(n+1)!}{n!} = \frac{n+1}{|\alpha-n|} \xrightarrow[n\to\infty]{} 1\,(=r).$$

(b) $\sum_{m\geq 0} \frac{(3m)!}{(m!)^3} t^{3m}$ is a function that arises from integrals on the cubic curve $C_t = \{xy = t(x^3+y^3+1)\} \subseteq \mathbb{C}^2$. Thinking of this as a power series in t^3, the ratio test gives:

$$r^3 = \lim_{m\to\infty} \frac{(m+1)^3}{(3m+3)(3m+2)(3m+1)} = \frac{1}{3^3} \implies r = \frac{1}{3}.$$

This has geometric meaning: $\frac{1}{3}$ is the smallest nonzero value of t for which the curve C_t is singular.