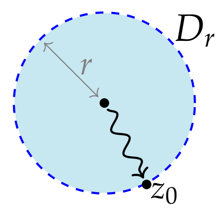
I.E. Variations on Abel's theorem

§ Theme: Abel's theorem (and summation by parts).

Given $\{a_k\} \subseteq \mathbb{C}$, we set (and assume positive)

$$r := \text{lub}\left\{ s \in \mathbb{R}_{\geq 0} \;\middle|\; \sum |a_k| s^k \text{ converges} \right\}.$$

Then $\sum a_k z^k$ converges *absolutely* for $|z| < r$, and *uniformly* for $|z| \leq s < r$ (to a continuous function, by Thm. I.D.4(ii)), hence defines a continuous function on D_r. Call this $f(z)$.



To what extent is an extension to $\overline{D}_r$ defined and continuous? We'll start by proving a result to the effect that for any $z_0 \in \partial D_r$ (i.e. $|z_0| = r$) with $\sum a_n z_0^n$ *convergent*, the limit of $f(z)$ along the radial path out to z_0 equals $\sum a_n z_0^n$. (We will refine this result later.) In fact, it suffices to consider $r = 1$ and $z_0 = 1$, and work on the real line. (Note that the $\{a_n\}$ are still very much complex numbers!) Set

$$A := \sum_{k \geq 0} a_k,$$

which we assume converges, and define

$$f(x) := \sum_{k \geq 0} a_k x^k$$

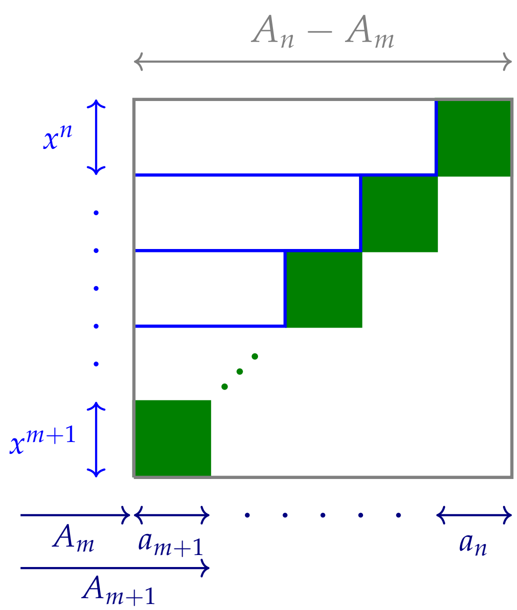
for $x \in [0, 1]$. From the above, *we know that f is continuous on* $[0, 1)$.

I.E.1. THEOREM (Abel, 1826). $\lim_{x \to 1^-} f(x) = A$, *i.e. f is continuous on* $[0, 1]$.

PROOF. Set $A_n := \sum_{k=0}^{n} a_k$. This sequence is Cauchy since it converges. The sequence of functions $f_n(x) := \sum_{k \geq 0}^{n} a_k x^k$ converges pointwise to $f(x)$ for $x \in [0, 1]$. Now (for $x \in [0, 1]$)

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| \textstyle\sum_{k=m+1}^{n} a_k x^k \right| \\ &= \left| (A_n - A_m) x^n + \textstyle\sum_{k=m+1}^{n-1} (A_k - A_m)(x^k - x^{k+1}) \right| \\ &\leq \epsilon + \epsilon \textstyle\sum_{k=m+1}^{n-1} (x^k - x^{k+1}) \quad \text{[taking } n, m \text{ suff. large]} \\ &\leq 2\epsilon \quad \text{[since the sum collapses]}, \end{aligned}$$

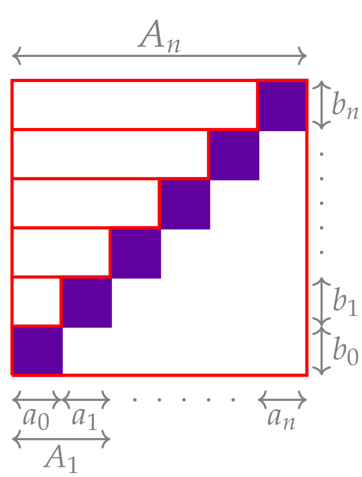
where we used $x^k - x^{k+1} \geq 0 \implies |x^k - x^{k+1}| = x^k - x^{k+1}$. The second equality can be understood visually from the figure:



So we have shown that $\{f_n\}$ is Cauchy in $\|\cdot\|_{[0,1]}$. We conclude that $\{f_n\}$ converges uniformly on $[0,1]$ to its pointwise limit, which is f. Therefore, by Theorem I.D.4(ii), f is continuous. $\square$

We draw attention to the principle (used in the proof, and further explored in the Exercises) of **summation by parts**: given sequences $\{a_k\}$ and $\{b_k\}$ with $A_n := \sum_{k=0}^{n} a_k$, we have

(I.E.2)
$$\sum_{k=0}^{n} a_k b_k = A_n b_n + \sum_{k=0}^{n-1} A_k(b_k - b_{k+1}).$$



§ Variation 1: Cauchy's (other) Theorem.

Let

- $\{a_n\}$ be a sequence of complex numbers with $\limsup\limits_{n\to\infty} |a_n|^{1/n} = 1$
- $s_n := \sum_{k=0}^{n} a_k$ be the sequence of nth partial sums
- $\sigma_n := \frac{1}{n}(s_0 + s_1 + \cdots + s_{n-1})$ the sequence of "Cesàro sums"
- $f(x) := \sum_{n\geq 0} a_n x^n$ the function on $(-1, 1) \subseteq \mathbb{R}$.

I.E.3. DEFINITION (Three conditions).

OS (ordinary summability) $\iff$ $\lim_{n\to\infty} s_n =: a$ exists
CS (Cesàro summability) $\iff$ $\lim_{n\to\infty} \sigma_n =: \sigma$ exists
AS (Abel summability) $\iff$ $\lim_{x\to 1^-} f(x) =: A$ exists

Note: I won't use "$f(1)$" here for $\sum a_n$ $(= a)$, but [**Ahlfors**] does this.

The point of these "variations" is to explore the relationship between these conditions and the resulting numbers a, σ, and A.

I.E.4. THEOREM (Cauchy, 1821). *OS ⇒ CS, with $\sigma = a$.*

PROOF. Let $\epsilon > 0$ be given. Pick $M \in \mathbb{N}$ such that

$$k \geq M \quad \Longrightarrow \quad |s_k - a| < \epsilon/2.$$

Let $B := \sup_{k\in\{0,\ldots,M-1\}} |s_k - a|$, and pick $N \geq M$ such that $\frac{MB}{N} < \frac{\epsilon}{2}$. Then $n \geq N \implies$

$$\begin{aligned} |\sigma_n - a| &= \frac{1}{n}\,|s_0 + \cdots + s_{n-1} - na| \\ &\leq \frac{1}{n}\sum_{k=0}^{n-1} |s_k - a| \\ &\leq \frac{MB}{N} + \frac{1}{n}\sum_{k=M}^{n-1} |s_k - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

§ Variation 2: Abel in the disk.

We begin by giving another proof of Abel's theorem.

I.E.5. THEOREM (Abel). *$OS \Rightarrow AS$, with $A = a$.*

PROOF. On $|x| < 1$ we can rearrange

$$\sum_{k\geq 0} s_k x^k = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots$$
$$= a_0 \sum_{k\geq 0} x^k + a_1 x \sum_{k\geq 0} x^k + a_2 x^2 \sum_{k\geq 0} x^k = \frac{1}{1-x} \sum_{n\geq 0} a_n x^n.$$

$\implies (1-x)\sum_{k\geq 0} s_k x^k = \sum_{n\geq 0} a_n x^n = f(x)$.

Since $\lim_{k\to\infty} s_k (= a)$ exists, given $\epsilon > 0$ we can take:

- M to be such that $k \geq M \implies |s_k - a| < \epsilon/2$;
- $B := \sup_{k\in\{0,\dots,M-1\}} |s_k - a|$; and
- $\delta > 0$ such that $1 - \delta < x < 1 \implies B(1 - x^M) < \epsilon/2$.

Then for $x \in (1-\delta, 1)$,

$$\begin{aligned} |f(x) - a| &= \left| (1-x)\sum_{k\geq 0} s_k x^k - (1-x)\sum_{k\geq 0} a x^k \right| \\ &= \left| (1-x)\sum_{k\geq 0} (s_k - a) x^k \right| \\ &\leq (1-x)\sum_{k=0}^{M-1} |s_k - a| x^k + (1-x)\sum_{k\geq M} |s_k - a| x^k \\ &\leq B(1-x^M) + x^M \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

where we used that $\sum_{k=0}^{M-1} = \frac{1-x^M}{1-x}$. □

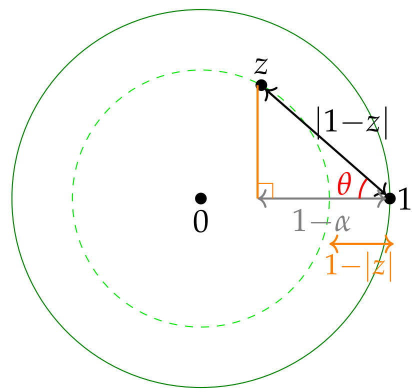
Discussion: What has to be done to make this proof viable inside the unit disk? Everything is fine up to

$$\begin{aligned} \left| (1-z)\sum_{k\geq 0} (s_k - a) z^k \right| &\leq |1-z| \sum_{k=0}^{M-1} |s_k - a||z|^k + |1-z| \sum_{k\geq M} |s_k - a||z|^k \\ &\leq B(1 - |z|^M)\frac{|1-z|}{1-|z|} + \frac{\epsilon}{2}|z|^M \frac{|1-z|}{1-|z|}, \end{aligned}$$

where $\frac{|1-z|}{1-|z|}$ causes problems if you just take $|z| < 1$ and $|1-z| < \delta$. We need to assume that

$$\frac{|1-z|}{1-|z|} \leq K \tag{I.E.6}$$

for some fixed $K \in \mathbb{R}_{>0}$. Note that for $z = \alpha + \mathbf{i}\beta$,



$$\frac{|1-z|}{1-|z|} \leq K \implies \frac{|1-z|}{1-\alpha} \leq K$$

using $\alpha \leq |z| \implies 1-\alpha \geq 1-|z| \implies \frac{1}{1-\alpha} \leq \frac{1}{1-|z|}$. In the diagram we find that $\cos(\theta) = \frac{1-\alpha}{|1-z|}$, and so (I.E.6) implies $\cos(\theta) \geq K^{-1}$ or equivalently

$$|\theta| \leq \arccos(K^{-1}). \tag{I.E.7}$$

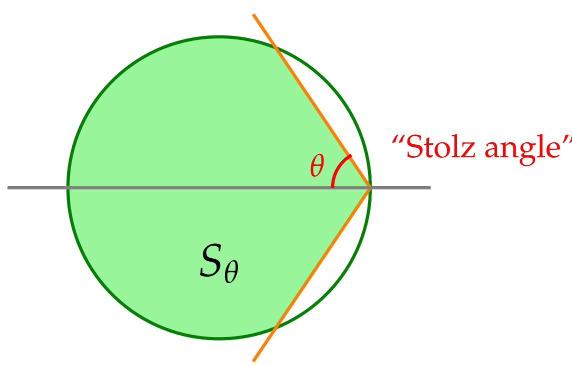
In the limit $z \to 1$, it turns out that (I.E.6) and (I.E.7) are equivalent.

Having fixed the bound K, we proceed as before and obtain that $|f(z) - a| < K\epsilon$, yielding:

I.E.8. THEOREM (Abel in the disk). *If $\sum a_n (= a)$ is convergent, then*

$$\lim_{S_\theta \ni z \to 1} f(z) = a$$

for any $\theta < 90°$.



That is, you can fix $\theta = 89.99°$, but then your path of approach must be in $S_{89.99°}$, or the limit you get may not equal a (or may not exist).

§ **Variation 3: a theorem of Frobenius**.

The Abel and Cauchy theorems say OS is stronger than CS or AS. But which one is weakest?

I.E.9. THEOREM (Frobenius, 1880). *CS $\implies$ AS, with $A = \sigma$. So OS $\implies$ CS $\implies$ AS.*

PROOF. For $|x| < 1$, we can write

$$\begin{aligned}\sum_{n\geq 0} a_n x^n &= s_0 + \sum_{n\geq 1}(s_n - s_{n-1})x^n \\ &= \sum_{n\geq 0} s_n x^n - \sum_{n\geq 0} s_n x^{n+1} \\ &= (1-x)\sum_{n\geq 0} s_n x^n \\ &= (1-x)^2 \sum_{n\geq 0}(s_0 + \cdots + s_n)x^n\end{aligned}$$

where we can rearrange in this fashion (as in the proof of Thm. I.E.5) since $\limsup |s_n|^{1/n} = 1$. Continuing, and using $\frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n\geq 0} x^n = \sum_{n\geq 0}(n+1)x^n$, this

$$\begin{aligned}&= (1-x)^2 \sum_{n\geq 0}(n+1)\sigma_{n+1}x^n \\ &= \frac{\sum(n+1)\sigma_{n+1}x^n}{\sum(n+1)x^n} \\ &= \sigma + \frac{\sum(n+1)(\sigma_{n+1}-\sigma)x^n}{\sum(n+1)x^n} \\ &= \sigma + (1-x)^2\sum_{n\geq 0}(n+1)(\sigma_{n+1}-\sigma)x^n.\end{aligned}$$

Taking $x \in (1-\delta, 1)$, we can bound the modulus of the second term, in analogy to the proof of Thm. I.E.5, by

$$B(1-x)^2\sum_{n=0}^{M-1}(n+1)x^n + (1-x)^2\sum_{n\geq M}(n+1)\frac{\epsilon}{2}x^n$$

$$< B\cancel{(1-x)^2}\frac{1-(M+1)x^M + Mx^{M+1}}{\cancel{(1-x)^2}} + \frac{\epsilon}{2}x^M$$

which is $< \frac{\epsilon}{2} + \frac{\epsilon}{2}$ for $M \gg 0$. $\square$

§ Coda: some examples.

I.E.10. EXAMPLE. Let $a_n = (-1)^n(n+1)$. We have

$$\begin{aligned}\lim_{x\to 1^-}\sum(-1)^n(n+1)x^n &= \lim_{x\to(-1)^+}\sum(n+1)x^n\\ &= \lim_{x\to(-1)^+}\frac{1}{(1-x)^2}=\frac{1}{4},\end{aligned}$$

and so AS holds. But $\sigma_{2n+1}\approx\frac{1}{2}$ and $\sigma_{2n}=0$, so CS fails.

I.E.11. EXAMPLE. Now consider $a_n = (-1)^n$. Clearly OS fails. But

$$s_n=\begin{cases}1, & n \text{ even}\\ 0, & n \text{ odd}\end{cases} \implies \sigma_n=\begin{cases}\frac{1}{2}, & n \text{ even}\\ \frac{1}{2}+\frac{1}{2n}, & n \text{ odd}\end{cases}\xrightarrow{n\to\infty}\frac{1}{2},$$

so CS holds. As Thm. I.E.9 says, the Abel sum equals the Cesaro sum:

$$\lim_{x\to 1^-}\sum(-1)^n x^n = \lim_{x\to(-1)^+}\sum x^n = \lim_{x\to -1}\frac{1}{1-x}=\frac{1}{2}.$$

I.E.12. EXAMPLE. Finally, we look at $a_n = 1/n^k$ ($a_0 = 0, k\geq 2$). In fact, $\sum\frac{1}{n^k}=\zeta(k) \implies$ OS. Note that

$$\sum_{n\geq 1}\frac{x^n}{n^k} =: \mathrm{Li}_k(x)$$

defines the k^{th} **polylogarithm** function, and (by Abel) this must have

$$\lim_{x\to 1^-}\mathrm{Li}_k(x)=\zeta(k). \tag{I.E.13}$$

Altogether we get the table:

Example	OS	CS	AS
$(-1)^n(n+1)$	$\times$	$\times$	✓
$(-1)^n$	$\times$	✓	✓
$1/n^k$	✓	✓	✓

which shows all our implications are "sharp".

§ Epilogue: Tauber's theorem.

The story now seems complete, in that we've proved the implications OS $\implies$ CS $\implies$ AS, and shown by examples that they are strict (i.e. their converses fail). What's missing is a result allowing us to deduce information about the sequence $\{a_n\}$ (its asymptotic behavior, or its ordinary summability) from the limiting behavior of the function $f(z) = \sum_{n\geq 0} a_n z^n$ as $|z| \to 1$. This is crucial in many parts of mathematics, not just analysis; and yet our results suggest that won't work because the converse of "OS $\implies$ AS" is false.

But that would be to overlook the possibility of *partial* converses! In fact there is a whole of analysis, *Tauberian theory*, which provides this kind of information, and its first result is

I.E.14. THEOREM (Tauber, 1897). *$AS + na_n \to 0 \implies OS$.*

PROOF. For $x \in (0,1)$ we can use

$$1 - x^n = (1-x)(1 + x + \cdots + x^{n-1}) \leq (1-x)n$$

to write

$$\begin{aligned}\left|\left(\sum_{n=0}^{N} a_n\right) - f(x)\right| &= \left|\sum_{n=1}^{N} a_n(1-x^n) - \sum_{n\geq N+1} a_n x^n\right| \\ &\leq \sum_{n=1}^{N}(1-x)n|a_n| + \frac{1}{N}\sum_{n\geq N+1}|na_n|x^n \\ &\leq (1-x)\sum_{n=1}^{N}|na_n| + \frac{1}{N(1-x)}\sup_{n\geq N}|na_n|.\end{aligned}$$

Therefore (taking $x = 1 - \frac{1}{N}$):

$$\left|s_N - f\left(1 - \frac{1}{N}\right)\right| \leq \frac{1}{N}\sum_{n=1}^{N}|na_n| + \sup_{n\geq N}|na_n|. \tag{I.E.15}$$

Now $\lim_{n\to\infty} na_n = 0 \implies \lim_{N\to\infty}$ RHS(I.E.15) $= 0$, while AS $\implies$ $\lim_{N\to\infty} f(1-\frac{1}{N}) =: A$ exists. It follows that $\lim_{N\to\infty} s_N = A$; in other words, $\sum a_n$ exists. □

To complete our discussion of these results, we contrast what Abel & Tauber say for a function $f(x) := \sum a_n x^n$ on $(-1, 1)$:

- Abel: $\sum a_n$ converges $\implies$ f has continuous extension to $(-1, 1]$ (by setting $f(1) := \sum a_n$).
- Tauber: f has continuous extension to $(-1, 1]$ (and $na_n \to 0$) $\implies$ $\sum a_n$ converges (to $f(1)$).

So Tauber's theorem is a **conditional converse**. There's a stronger version due to Littlewood (1911), relaxing "$na_n \to 0$" to "$|na_n| \leq B$".

Remark: $a_n = \frac{1}{n \log(n)}$ ($n \geq 2$) is an example of a sequence satisfying $na_n \to 0$ but *not* ordinary summable: $\sum_{n \geq 2} \frac{1}{n \log n} = \infty$, essentially by the integral test:

$$\int_2^\infty \frac{dx}{x \log x} = \int_{\log 2}^\infty \frac{du}{u} = \infty.$$

So we really do need *both* hypotheses in Tauber's Theorem to conclude OS.

Exercises

(1) Let $\{b_n\}$ be a decreasing sequence of positive (real) numbers approaching zero. Prove that the power series $\sum b_n z^n$ is uniformly convergent on the domain of z such that $|z| \leq 1$ and $|z - 1| \geq \delta$, where $\delta > 0$. [Hint: use summation by parts in the form $\sum_{k=m+1}^n a_k b_k = (A_n - A_m) b_n + \sum_{k=m+1}^{n-1} (A_k - A_m)(b_k - b_{k+1})$, with $a_n = z^n$.]

(2) Determine whether $a_n = \cos(n)$ is summable in each of the three senses in this Chapter (AS, CS, OS).

(3) Show that if $\{a_n\} \subset \mathbb{C}$ has bounded partial sums and $\{b_n\} \subset \mathbb{R}$ is a nonincreasing sequence converging to zero, then $\sum a_n b_n$ converges.

(4) Prove that $\sum_{n \geq 1} \frac{\sin(n\theta)}{n^\alpha}$ converges for $\alpha > 0$ and $\theta \notin 2\pi\mathbb{Z}$. [Hint: apply (3).]