I.F. Analytic functions

§ Definition and basic properties.

Let $U \subseteq \mathbb{C}$ be open, and $f \in \mathcal{F}(U)$ and $\mathbb{C}$-valued function.

I.F.1. DEFINITION. (i) f is **analytic at $z_0 \in U$** if there exist a real number $r \in \mathbb{R}_{>0}$ and a sequence $\{a_n\} \subseteq \mathbb{C}$ such that $\sum_{n\geq 0} a_n(z-z_0)^n$ converges absolutely to $f(z)$ for all $z \in D(z_0, r)$. That is, *f is described by a power series expansion in a neighborhood of z_0*, or simply "*f has a power series expansion at z_0*".

(ii) f is **analytic on U** (written $f \in \mathscr{A}n(U)$) if f is analytic at every $z_0 \in U$.

We can also define analyticity on an arbitrary set $S \subseteq \mathbb{C}$. A given $f \in \mathcal{F}(S)$ is "analytic on S" if there is an open set $U \subseteq \mathbb{C}$ which contains S and $F \in \mathscr{A}n(U)$ such that $f \equiv F|_S$. In other words, *f must extend to an analytic function on an open set containing S*.

As you would expect, $\mathcal{A}n(U)$ is a $\mathbb{C}$-algebra:

I.F.2. PROPOSITION. *Given $f, g \in \mathscr{A}n(U)$, we have $f+g, fg, \alpha f \in \mathscr{A}n(U)$ and $f/g \in \mathscr{A}n(U \setminus \{g=0\})$.*

PROOF. Given $z_0 \in U$, f and g are represented (on $D(z_0, r)$) by η_r of two formal power series, for some sufficiently small $r > 0$. (Here we modify η_r to send $T \mapsto z - z_0$.) The *formal* sum and product converge on the same disk, to $f+g$ and fg. Assuming that $g(z_0) \neq 0$ (equivalently $\mathrm{ord}(g(T)) = 0$), the formal quotient converges on a possibly smaller disk to f/g. See Theorem I.D.23(ii) and Proposition I.D.26. That we can do this at each $z_0 \in U$ ensures that $f+g, fg, f/g$ are analytic where they are defined. □

I.F.3. PROPOSITION. *$f \in \mathscr{A}n(U)$, $g \in \mathscr{A}n(V)$, $g(V) \subset U \implies f \circ g \in \mathscr{A}n(V)$.*

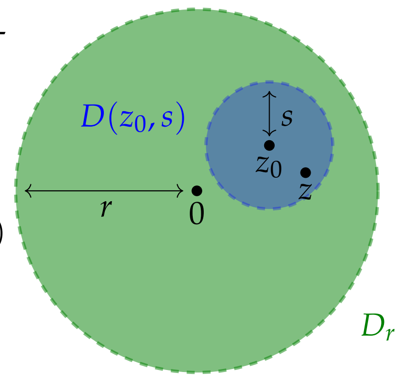
PROOF. Similar argument, using Proposition I.D.27. □

§ Relation to power series.

I.F.4. THEOREM. *Given a formal power series $f(T) \in \mathscr{P}_r$, the function $f(z) := \eta_r(f(T))$ belongs to $\mathscr{A}n(D_r)$.*

PROOF. We know that $f(z) \in \mathcal{C}^0(D_r)$ from Chapter I.D. It remains to show that f is analytic at any $z_0 \in D_r$.

Take $s < r - |z_0|$, so that $D(z_0, s) \subseteq D_r$ as in the picture. For $z \in D(z_0, s)$, by assumption we have



$$f(z) = \sum_{n \geq 0} a_n z^n,$$

which upon rewriting $z = z_0 + (z - z_0)$ and expanding becomes

$$\sum_{n \geq 0} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k.$$

The whole double sum converges absolutely since

$$\begin{aligned}\sum_{n \geq 0} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k} |z - z_0|^k &= \sum_{n \geq 0} |a_n| \left(|z_0| + |z - z_0|\right)^n \\ &\leq \sum_{n \geq 0} |a_n| (|z_0| + s)^n\end{aligned}$$

converges. (This is by hypothesis, since $f \in \mathscr{P}_r$ and $|z_0| + s < r$.) Switching order of summation is therefore permissible, and yields

$$f(z) = \sum_{k \geq 0} \left(\sum_{n \geq k} a_n \binom{n}{k} z_0^{n-k} \right) (z - z_0)^k,$$

the desired power series representation of $f(z)$ on $D(z_0, s)$. $\square$

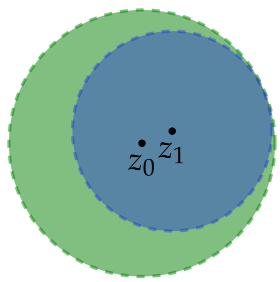
This establishes that η_r as defined in Chapter I.D factors through $\mathscr{A}n(D_r)$. We accordingly replace the codomain and write henceforth

$$\eta_r \colon \mathscr{P}_r \to \mathscr{A}n(D_r).$$

§ Zeroes of analytic functions.

I.F.5. THEOREM. *Assume $f, g \in \mathscr{An}(U)$ with U connected open. If the set $\mathrm{acc}\{z \in U \mid f(z) = g(z)\}$ is nonempty, then $f \equiv g$ on U.*

PROOF. From the last proof, if the (unique) power series representing a function F at z_0 has radius of convergence $r_F(z_0) < \infty$, then the radius $r_F(z_1)$ for z_1 near to z_0 cannot be very different.



Indeed, the argument shows that if F is represented by a power series (about z_0) on the green disk of radius r, then it is represented by a power series about z_1 on a disk of radius at least $r - |z_0 - z_1|$.[18] So

$$r_F(z_0) - r_F(z_1) \leq |z_0 - z_1|$$

and as the argument is clearly symmetric,

$$|r_F(z_0) - r_F(z_1)| \leq |z_0 - z_1|.$$

Hence the "radius of convergence function"

$$r_F \colon U \to \mathbb{R}_{>0} \cup \{\infty\}$$

is either a positive continuous function or everywhere infinity.

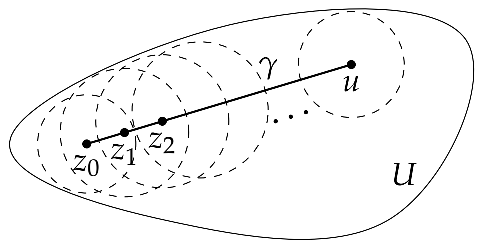
Let $z_0 \in \mathrm{acc}\{z \in U \mid f(z) = g(z)\}$, write $F = f - g$, and let $u \in U$ be arbitrary. We want to show $F(u) = 0$. Because U is connected, there is a path γ from z_0 to u. Since γ is compact and r_F is continuous,

[18]The argument allows any radius $s < r - |z_0 - z_1|$, but then we can take "=" since radius of convergence is always the least upper bound of the set of radii of convergence disks.

the restriction $r_F|_\gamma$ attains a minimum value[19] $b \in \mathbb{R}_{>0}$, i.e.

$$r_F(z) \geq b\,(> 0) \text{ for every } z \in \gamma.$$

Let $\nu < \frac{b}{2}$ and take $z_1, z_2, z_3, \ldots$ within ν of each other. The neighborhoods of convergence about each z_i therefore contain z_{i+1}, z_{i-1}.



Recall that in the situation where F is represented by power series at z_0 with $z_0 \in \operatorname{acc}\{z \mid F(z) = 0\}$, that power series is identically zero (see Theorem I.D.23(iii)). This implies

$$F \equiv 0 \quad \text{on} \quad D(z_0, b).$$

Since $z_1 \in D(z_0, b)$, we have $z_1 \in \operatorname{acc}\{z \mid F(z) = 0\}$, which gives

$$F \equiv 0 \quad \text{on} \quad D(z_1, b).$$

This contains z_2; continuing the argument, we find $F(u) = 0$ as desired. $\square$

I.F.6. COROLLARY. *If U is connected open, $f \in \mathscr{An}(U)$ is not identically zero, and $K \subset U$ is compact, then f has only finitely many zeroes in K.*

PROOF. If f has infinitely many roots in K, they have a limit point in K (hence in U) by compactness (Theorem I.C.15). Applying Theorem I.F.5 with g the zero function yields a contradiction. $\square$

[19]This is just a slight variation on Theorem I.C.18.

§ <u>Relation with complex differentiability.</u>

The key result of this section is that analytic functions are holomorphic — in fact, infinitely complex differentiable. I want to put this in a broader context involving **formal differentiation**

$$\begin{aligned} D\colon \mathbb{C}[[T]] &\to \mathbb{C}[[T]] \\ \sum_{n\geq 0} a_n T^n &\mapsto \sum_{n\geq 1} n a_n T^{n-1} \end{aligned}$$

of formal power series. We denote the set of holomorphic functions on a region U by $\mathscr{Hol}(U)$.

I.F.7. THEOREM. *Let $f(T) \in \mathscr{P}_r \subset \mathbb{C}[[T]]$, with*

$$\eta_r(f(T)) =: f(z) \in \mathscr{An}(D_r);$$

then

(a) *Df has radius of convergence r, and*
(b) *$\frac{df}{dz}(z)$ exists and equals $\eta_r((Df)(T))$.*

Conclude that:

(c) *$\mathscr{P}_r$ is closed under D; and*
(d) *$\mathscr{An}(U) \subseteq \mathscr{Hol}(U)$, for any region U.*

PROOF. (a) Since the radius of convergence of $f(T)$ is at least r, we have[20] $\limsup |na_n|^{1/n} = \lim n^{1/n} \cdot \limsup |a_n|^{1/n} \leq \frac{1}{r}$. This shows that $(Df)(T) \in \mathscr{P}_r$ as desired. This also proves (c), which is simply the statement that $f \in \mathscr{P}_r \implies Df \in \mathscr{P}_r$. [N.B.: to show that $\lim_{n\to\infty} n^{1/n} = 1$, you can either take log and invoke L'Hôpital, or (to avoid such things) you may write

$$n = (1 + (n^{1/n} - 1))^n \ > \ 1 + \binom{n}{2}(n^{1/n} - 1)^2$$

$$\implies \ \cancel{n-1}\ 1 > \frac{n(\cancel{n-1})}{2}(n^{1/n} - 1)^2 \quad \implies \quad \sqrt{\frac{2}{n}} > n^{1/n} - 1 > 0$$

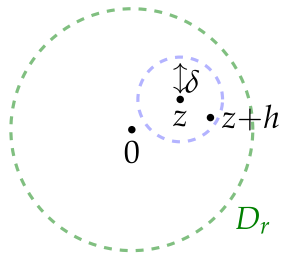
and apply the squeeze lemma.]

[20]It should be $|na_n|^{\frac{1}{n-1}}$; except one can multiply Df by T, and this of course converges wherever Df does.

(b) Recall that f is complex-differentiable at z with derivative α if

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h}-\alpha\right|=0. \tag{I.F.8}$$

Take $\delta < r-|z|$ and $|h|<\delta$ as shown in the picture:



then

$$\begin{aligned} f(z+h)-f(z) &= \sum_{n\geq 0} a_n((z+h)^n - z^n) \\ &= \left(\sum n a_n z^{n-1}\right) h + \left(\sum a_n P_n(z,h)\right) h^2 \end{aligned}$$

where

$$|P_n(z,h)| = \left|\sum_{k=2}^{n}\binom{n}{k}h^{k-2}z^{n-k}\right| \leq \sum_{k=2}^{n}\binom{n}{k}|h|^{k-2}|z|^{n-k}.$$

Reindexing by $\ell = k-2$ and using

$$\tfrac{n!}{k!(n-k)!} = n(n-1)\tfrac{(n-2)!}{(\ell+2)!(n-2-\ell)!} \leq n(n-1)\tfrac{(n-2)!}{\ell!(n-2-\ell)!},$$

this becomes

$$\begin{aligned} |P_n(z,h)| &\leq n(n-1)\sum_{\ell=0}^{n-2}\binom{n-2}{\ell}\delta^{\ell}|z|^{n-\ell-2} \\ &= n(n-1)(|z|+\delta)^{n-2} = n(n-1)r_0^{n-2}, \end{aligned}$$

where $r_0 := |z|+\delta < r$. Hence,

$$\begin{aligned} \left|\frac{f(z+h)-f(z)}{h}-(Df)(z)\right| &= \left|\frac{f(z+h)-f(z)}{h}-\sum_{n\geq 1} n a_n z^{n-1}\right| \\ &= \left|h\sum a_n P_n(z,h)\right| \\ &\leq |h|\sum |a_n| n(n-1) r_0^{n-2} \end{aligned}$$

where the last sum is convergent since $r_0 < r$ and $(n(n-1))^{1/n} \to 1$. Taking $h \to 0$ now gives (I.F.8) with $\alpha = (Df)(z)$.

(d) Let an analytic function $f \in \mathscr{An}(U)$ be given. For any point $z_0 \in U$, f is represented by a power series on some disk in U about z_0. By (b), f' exists on that neighborhood (and is represented by $\eta((Df)(T))$). So f' exists everywhere on U. $\square$

I.F.9. COROLLARY. (a) *Let $f \in \mathscr{An}(U)$; then f is infinitely complex-differentiable on U.*

(b) *Let $f \in \mathscr{An}(D_r)$ be given as $\eta_r(f(T))$. Then:*

(i) $f^{(k)}(z) = \eta(D^k f(T)) = k!a_k + h_k(z)$, where $h_k \in \mathscr{An}(D_r)$ with $h_k(0) = 0$. So $f^{(k)}(0) = k!a_k$.

*(ii) f has a **primitive** on D_r (i.e. F satisfying $F' = f$),*

$$F = \sum_{n\geq 0} \frac{a_n}{n+1} z^{n+1} (+constant) \in \mathscr{An}(D_r).$$

PROOF. (a) About each point f is represented by power series; and we just showed f' *is represented by the formal derivative* (in some neighborhood), *hence is analytic itself*. Therefore we may repeat the process indefinitely.

Part (i) of (b) is an immediate consequence of (a). Part (ii) follows from the Theorem and the fact that F has the same radius of convergence as f. $\square$

I.F.10. COROLLARY. *Suppose $f \in \mathscr{An}(U)$ with $|f|$ constant, Re(f) constant, Im(f) constant, or $f' \equiv 0$. Then f is constant.*

PROOF. We know that f is holomorphic, and so if $f' \equiv 0$ [resp. $|f|$ constant] the result follows from Prop. I.B.24 [resp. I.B Exercise (5)]. Now write $f = u + iv$. If $\mathrm{Re}(f) = u$ is constant, then $u_x = u_y = 0$, and by the Cauchy-Riemann equations $v_x = v_y = 0 \implies v =$ constant. The remaining case is similar. $\square$

§ Examples: exp and log.

Recall that we have holomorphic functions ($z = x + \mathbf{i}y$)

$$\exp(z) := e^x \cos(y) + \mathbf{i}e^x \sin(y) \in \mathscr{H}\!ol(\mathbb{C})$$
$$\log(z) := \log|z| + \mathbf{i}\arg(z) \in \mathscr{H}\!ol(\mathbb{C} \setminus \mathbb{R}_{\leq 0}),$$

which was proved by checking the Cauchy-Riemann equations.

Next write $f(T) := \log(1 - T) = -\sum_{n\geq 1} \frac{T^n}{n}$,

$$\widetilde{\exp}(z) := \eta_\infty\left(\sum_{n\geq 0} \frac{T^n}{n!}\right) = \eta_\infty(\exp(T)) \in \mathscr{A}\!n(\mathbb{C}), \text{ and}$$
$$\widetilde{\log}(1 - z) := \eta_1(f(T)) \in \mathscr{A}\!n(D_1);$$

we also had

$$\cos(z) := \eta_\infty(C(T)) = \eta_\infty\left(\sum_{k\geq 0} \frac{(-1)^k T^{2k}}{(2k)!}\right) \in \mathscr{A}\!n(\mathbb{C}) \text{ and}$$
$$\sin(z) := \eta_\infty(S(T)) = \eta_\infty\left(\sum_{l\geq 0} \frac{(-1)^l T^{2l+1}}{(2l+1)!}\right) \in \mathscr{A}\!n(\mathbb{C}).$$

The formal relation

$$\exp(\mathbf{i}T) = C(T) + \mathbf{i}S(T)$$

implies

$$\widetilde{\exp}(\mathbf{i}y) = \cos(y) + \mathbf{i}\sin(y) = \exp(\mathbf{i}y).$$

for $y \in \mathbb{R}$. By Calculus, we also have $\widetilde{\exp}(x) = \exp(x)$ for $x \in \mathbb{R}$.

Now in $\mathbb{C}[[T]]$, for any $\gamma_1, \gamma_2 \in \mathbb{C}$

$$\begin{aligned}
\exp((\gamma_1 + \gamma_2)T) &= \sum_{n\geq 0} \frac{(\gamma_1 T + \gamma_2 T)^n}{n!} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n} \frac{(\gamma_1 T)^k (\gamma_2 T)^{n-k}}{k!(n-k)!} \\
&= \left(\sum_{k\geq 0} \frac{(\gamma_1 T)^k}{k!}\right)\left(\sum_{l\geq 0} \frac{(\gamma_2 T)^l}{l!}\right) \\
&= \exp(\gamma_1 T)\exp(\gamma_2 T)
\end{aligned}$$

which applying η_∞ on both sides

$$\begin{aligned}
&\Longrightarrow \widetilde{\exp}((\gamma_1+\gamma_2)z) = \widetilde{\exp}(\gamma_1 z)\widetilde{\exp}(\gamma_2 z) \\
&\overset{z=1}{\Longrightarrow} \widetilde{\exp}(\gamma_1+\gamma_2) = \widetilde{\exp}(\gamma_1)\widetilde{\exp}(\gamma_2) \\
&\Longrightarrow \widetilde{\exp}(z) = \widetilde{\exp}(x)\widetilde{\exp}(\mathbf{i}y) = \exp(x)\exp(\mathbf{i}y) = \exp(z),
\end{aligned}$$

proving

$$\exp(z) \in \mathscr{An}(\mathbb{C}). \tag{I.F.11}$$

To do the same for log, by Calculus we know that

$$\widetilde{\log}(1-z) = \log(1-z) \quad \text{for } z = x \in (-1,1) \text{ real.}$$

Furthermore, $\exp(\log(1-x)) = 1-x$ there, and so the formal power series

$$(\exp\circ f)(T) - 1 + T \tag{I.F.12}$$

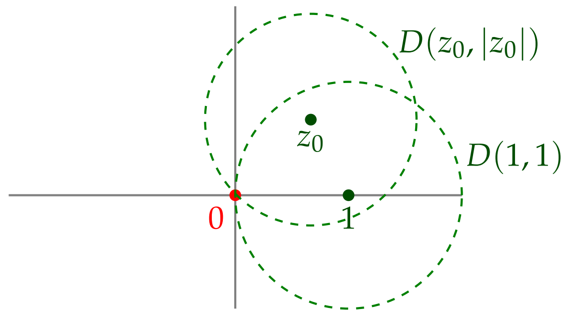
is sent by η to a function on D_1 which is identically zero on $(-1,1)$. Since $0 \in \operatorname{acc}\{(-1,1)\}$, we have from Theorem I.D.23 that (I.F.12) is the zero power series, and so

$$\exp(\widetilde{\log}(1-z)) = 1-z \quad \text{on } D_1.$$

We can repeat this (how?) for $\widetilde{\log}\circ\exp$ on some sufficiently small disk, and so

$$\begin{aligned}
\widetilde{\log}(z) &= \widetilde{\log}(|z|\exp(\mathbf{i}\arg z)) \\
&= \widetilde{\log}(\exp(\log|z| + \mathbf{i}\arg z)) \\
&= \log|z| + \mathbf{i}\arg(z) \\
&= \log(z)
\end{aligned}$$

on some neighborhood of $z=1$. *Hence* this is true on all of $D(1,1)$, as both functions are defined and have complex derivative $\frac{1}{z}$ there (and one can apply Prop. I.B.24).



We can do the same thing for $\log(z)$ on any disk $D(z_0,|z_0|)$ in $\mathbb{C}\setminus\mathbb{R}_{\le 0}$; if the center is z_0, write

$$\frac{1}{z}=\frac{\frac{1}{z_0}}{1+\frac{z-z_0}{z_0}}=\frac{1}{z_0}\left(1-\frac{1}{z_0}(z-z_0)+\frac{1}{z_0^2}(z-z_0)^2+\cdots\right)$$

integrate, and add $\log(z_0)$ as constant. If $z_0\in D(1,1)$, we have that the resulting $\widetilde{\widetilde{\log}}(z)$ agrees with $\widetilde{\log}(z)$ at z_0 and has the same derivative $\frac{1}{z}$, hence by the above properties of analytic functions must agree on $D(1,1)\cap D(z_0,|z_0|)$. In this way one shows that

$$\log(z)\in\mathscr{An}(\mathbb{C}\setminus\mathbb{R}_{\le 0}). \tag{I.F.13}$$

Finally, from

$$\exp(\log\alpha\beta-\log\alpha-\log\beta)=\frac{\exp(\log\alpha\beta)}{\exp(\log\alpha)\exp(\log\beta)}=\frac{\alpha\beta}{\alpha\cdot\beta}=1$$

we get

$$\log(\alpha\beta)\equiv\log\alpha+\log\beta\pmod{2\pi\mathbf{i}\mathbb{Z}}.$$

Out of this, we can produce even more analytic functions:

I.F.14. EXAMPLE. By composing analytic functions,

$$z^\alpha:=\exp(\alpha\log(z))\in\mathscr{An}(\mathbb{C}\setminus\mathbb{R}_{\le 0}).$$

Here, $\mathbb{R}_{\le 0}$ can be replaced by *any* slit going from 0 to ∞, e.g. $\mathbb{R}_{\ge 0}$. Alternatively, instead of deleting the slit, you can think of z^α as being ambiguous by $\{\exp(2\pi\mathbf{i}\alpha\mathbb{Z})\}$, which consists of finitely many values if and only if $\alpha\in\mathbb{Q}$ (see the discussion of Riemann surfaces below).

I.F.15. EXAMPLE. Since $\cos(z)$ and $\sin(z)$ belong to $\mathscr{An}(\mathbb{C})$ by construction, any identity which is true on the level of power series holds also for the functions — in particular,

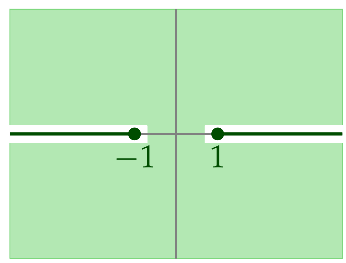
$$\cos(z) = \tfrac{1}{2}(\exp(\mathbf{i}z) + \exp(-\mathbf{i}z)).$$

Writing $w := \cos(z)$ and $\varepsilon = \exp(\mathbf{i}z)$, this reads

$$\begin{aligned} 2w = \varepsilon + \varepsilon^{-1} \implies & 0 = \varepsilon^2 - 2w\varepsilon + 1 \\ \implies & \varepsilon = \frac{2w \pm \sqrt{4w^2-4}}{2} = w \pm \sqrt{w^2-1} \\ \implies & \arccos(w) = z = -\mathbf{i}\log(\varepsilon) \\ & \qquad = -\mathbf{i}\log\left(w \pm \sqrt{w^2-1}\right). \end{aligned}$$

Choosing the "+" branch, $w + \sqrt{w^2-1}$ is an analytic function on the region

$$\mathbb{C} \setminus (\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1})$$



because $\sqrt{u} = u^{1/2}$ is of the form in the previous example, and $w^2 - 1$ maps this region into $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Furthermore, considering the equations

$$\begin{aligned} w + \sqrt{w^2-1} = r \in \mathbb{R} \quad & \implies \\ \sqrt{w^2-1} = r - w \quad & \implies \\ \cancel{w^2} - 1 = r^2 - 2rw + \cancel{w^2} \quad & \implies \\ w = \frac{r^2+1}{2r} = \frac{1}{2}(r^{-1} + r) \in \mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq -1} \cup \{\infty\} & \end{aligned}$$

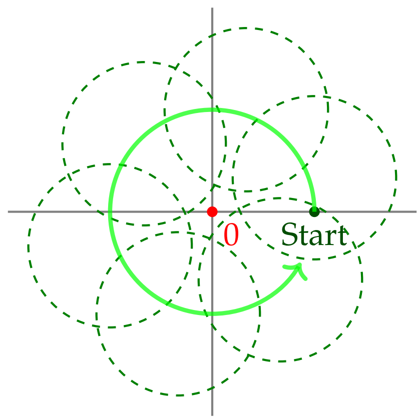
we see that $w + \sqrt{w^2-1}$ maps the region $\mathbb{C} \setminus (\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1})$ into $\mathbb{C} \setminus \mathbb{R}$ — in fact, into $\mathfrak{H}$ — where we can also define $\log(\cdot)$ as an analytic function. It follows that the composition is analytic:

$$\arccos(w) \in \mathscr{An}(\mathbb{C} \setminus (\mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1})).$$

I.F.16. REMARK. We have to do this all by hand, in the absence of something like the inverse mapping theorem (which wouldn't give us analyticity of arccos on more than a disk anyway!).

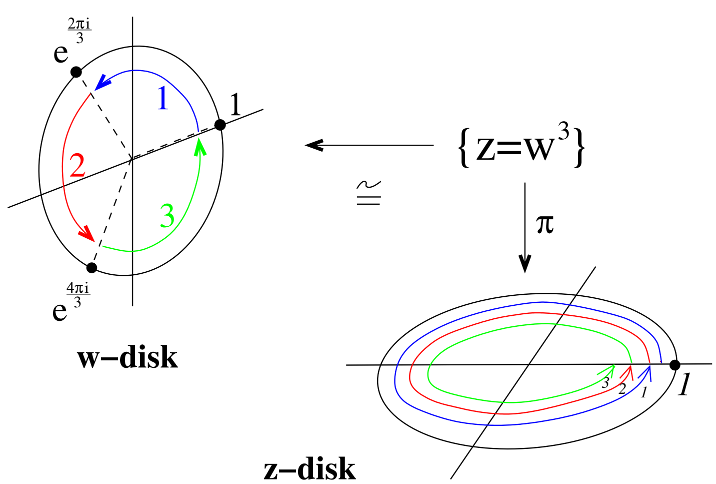
§ <u>An overview of Riemann surfaces</u>.

Let's have another look at the z^α example — taking (say) $\alpha = \frac{1}{3}$. As alluded to above, if we keep pasting power-series together along overlaps of neighborhoods arranged around the origin,



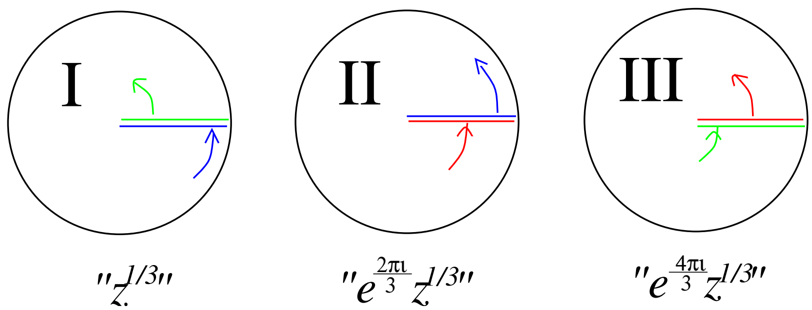
we quickly run into "multivalued" behavior in the functions constructed from log. So the question arises as to what the "existence domain" of a multivalued function over an open set in $\mathbb{C}$ (i.e. the minimal "cover" of U making the function single-valued) looks like. The resulting "complex 1-manifolds" are called **Riemann Surfaces**.

I.F.17. EXAMPLE. Let's visualize the "Riemann surface of $(w{=})\, z^{\frac{1}{3}}$ over the unit disk". This is some object fitting (as "$\{z{=}w^3\}$") into the following picture:

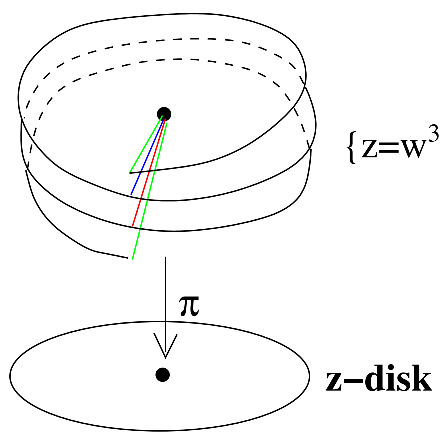


To construct it, think about following $z^{\frac{1}{3}}$ around the disk once counterclockwise: when you reach your starting point the function has become $e^{\frac{2\pi i}{3}}$ times the branch of $z^{\frac{1}{3}}$ you started with; going around once more, you get $e^{\frac{4\pi i}{3}}z^{\frac{1}{3}}$; and one more time gets you back to your original branch.

So taking three unit disks, slitting them along the positive reals, and gluing them as indicated



we get the "cyclic parking lot"

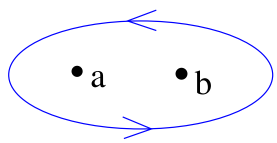


(The green segments are glued but, alas, I can't draw in 4 dimensions.) An easier way to visualize this Riemann surface is more tautological: it's just the w-disk. The difficulty is in seeing the w-disk "over" the z-disk.

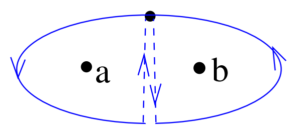
I.F.18. EXAMPLE. Next, let's construct an existence domain for

$$\mathfrak{F}(z) = \sqrt{(z-a)(z-b)(z-c)}$$

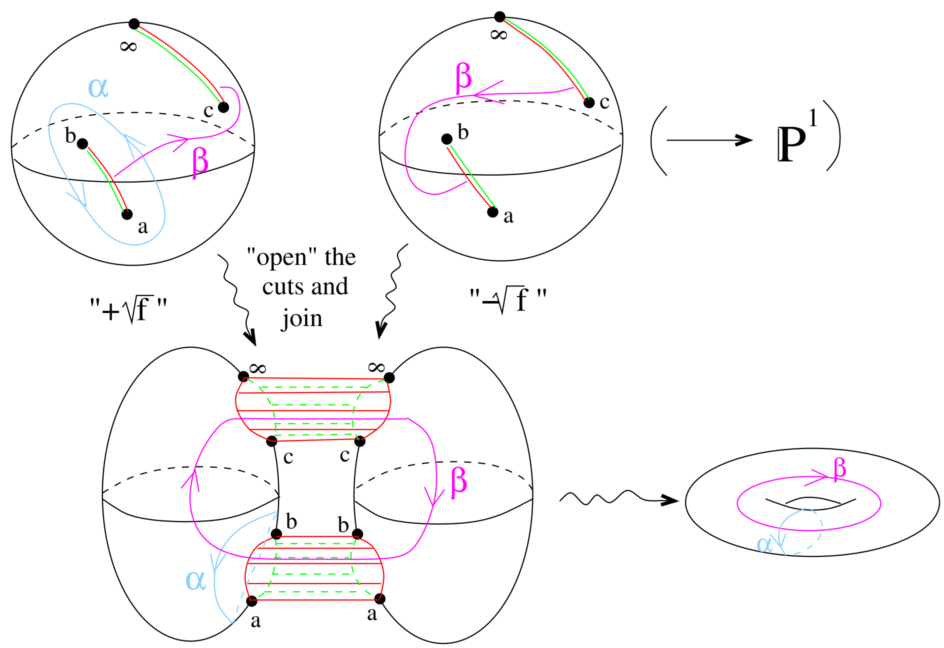
over $\hat{\mathbb{C}}$ (or $\mathbb{C} \setminus \{a,b,c\}$ if you prefer). In a neighborhood of $z_0 = a, b, c$ this looks like the "Riemann surface of $(z-z_0)^{\frac{1}{2}}$ over a disk", which is the same as the construction we just did except with 2 unit disks instead of 3. Indeed, going once around a, b, or c takes $\mathfrak{F} \mapsto -\mathfrak{F}$; and furthermore, because the degree of the polynomial under the square root is odd, going once around ∞ does the same thing. Since



is equivalent to



going around two points at once gives no change. So taking two $\hat{\mathbb{C}}$'s and cutting and pasting them as indicated on the next slide, we end up with a manifold of genus 1 on which $\mathfrak{F}$ becomes well-defined:



In the picture, α and β are called 1-cycles; they are just there to make the topology clear. The same construction works if we replace $\mathfrak{F}(z)$ by $\sqrt{(z-a)(z-b)(z-c)(z-d)}$, with d replacing ∞.

See **[Ahlfors]** for more examples of existence domains/Riemann surfaces:

- $\log(z)$ (∞ level parking lot)
- $\arccos(z)$ (more complicated).

Exercises

(1) For any positive integer k, find a nontrivial formal power series satisfying the Bessel equation $\{T^2D^2 + TD + (T^2 - k^2)\}f(T) = 0$, where D is the the formal differentiation operator. (Take the first nonvanishing term to be $\frac{T^k}{k!2^k}$.) What is its radius of convergence? Conclude that η of your formal power series is an analytic solution to the corresponding ordinary differential equation.

(2) Suppose that[21] $f \in \eta_r(\mathscr{P}_r) \subseteq \mathscr{A}n(D_r)$ with $r > 1$. Show that there exists $C > 0$ such that $|f^{(n)}(0)| \leq C \cdot n!$ $(\forall n)$.

(3) Determine all values of $2^{\mathbf{i}}$ and $\mathbf{i}^{\mathbf{i}}$.

(4) Find the real and imaginary parts of (a) z^z and (b) $\cos(x + \mathbf{i}y)$ [use addition formulas].

(5) Describe the Riemann surface associated to $z = \frac{1}{2}(w + \frac{1}{w})$ (i.e. the existence domain of $w(z)$).

(6) Find open sets where (a) $\sqrt{1+z} + \sqrt{1-z}$ and (b) $\log(\log(z))$ are analytic.

(7) Express arctan in terms of log (refer to the treatment of arccos above) and write down a maximal set where it is analytic.

(8) Let $f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$. Show that $f'(z) = \frac{1}{z^2+1}$, and conclude that f represents arctan on the unit disk.

[21] We do not yet know, although this will turn out to be true, that all of $\mathscr{A}n(D_r)$ is in the image of η_r.