I.G. Fractional linear transformations

§ Group structure.

In this course we'll meet the **automorphism groups** (of 1-to-1 analytic self-maps)

$$\begin{aligned}\mathrm{Aut}(\mathbb{C}) &= \{z \mapsto \alpha z + \beta \mid \alpha, \beta \in \mathbb{C}, \alpha \neq 0\} \\ \mathrm{Aut}(D_1) &= \left\{ z \mapsto e^{\mathbf{i}\varphi} \frac{z-\alpha}{1-\bar{\alpha}z} \;\middle|\; \varphi \in \mathbb{R},\ \alpha \in D_1 \right\} \\ \mathrm{Aut}(\mathfrak{H}) &= \left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R},\ ad-bc=1 \right\}\end{aligned}$$

as well as the isomorphism

$$\begin{aligned}\mathfrak{H} &\xrightarrow{\cong} D_1 \\ z &\longmapsto \frac{z-\mathbf{i}}{z+\mathbf{i}}\end{aligned}$$

All are **fractional linear transformations** (abbreviated "FLT")

$$f_M(z) := \frac{az+b}{cz+d}, \qquad \text{with } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}), \tag{I.G.1}$$

i.e. $a,b,c,d \in \mathbb{C}$ and $ad-bc \neq 0$.

I.G.2. PROPOSITION-DEFINITION. *The fractional linear transformations form a group* FLT *under composition of functions, and*

$$M \longmapsto f_M(z)$$

defines a surjective group homomorphism

$$GL_2(\mathbb{C}) \twoheadrightarrow \underline{\mathrm{FLT}} \tag{I.G.3}$$

with kernel the matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, $a \in \mathbb{C}^*$.

PROOF. First, if $N = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then

$$\begin{aligned}(f_M \circ f_N)(z) &= \frac{a(\frac{Az+B}{Cz+D})+b}{c(\frac{Az+B}{Cz+D})+d} = \dots \\ &= \frac{(aA+bC)z+(aB+bD)}{(cA+dC)z+(cB+dD)} = f_{M\cdot N}(z).\end{aligned}$$

This gives the homomorphism property, and surjectivity holds by definition. For the kernel, suppose f_M is the identity transformation, so that $\frac{az+b}{cz+d} = z$ ($\forall\, z \in \mathbb{C}$). Then

$$\begin{aligned} & 0 = cz^2 + (d-a)z - b \ (\forall z) \\ \implies\ & c = (d-a) = b = 0 \\ \implies\ & M = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right). \end{aligned}$$

Conversely, if M is of this form, then $f_M(z) = \frac{az+0}{0z+a} = z$. □

So in fact (I.G.3) factors through an *isomorphism* from

$$\text{(I.G.4)} \qquad \mathrm{PGL}_2(\mathbb{C}) := \mathrm{GL}_2(\mathbb{C}) / \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \middle|\ a \in \mathbb{C}^* \right\} \cong \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{id}\}$$

to $\underline{\text{FLT}}$, by Prop. I.G.2 and the fundamental theorem of group homomorphisms. That is, we have identified $\underline{\text{FLT}}$ as a group.

But what are the FLT's really transformations *of*? We can view f_M as a function from

$$\mathbb{C} \setminus \{-\tfrac{d}{c}\} \xrightarrow{\cong} \mathbb{C} \setminus \{\tfrac{a}{c}\}$$

which may be extended to

$$\hat{\mathbb{C}} \xrightarrow{\cong} \hat{\mathbb{C}}$$

by sending $-\frac{d}{c} \mapsto \infty$ and $\infty \mapsto a/c$.

I.G.5. REMARK. If we consider $f \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that

(i) $z \mapsto f(z)$ induces an analytic isomorphism from $\mathbb{C} \setminus \{\alpha\}$ to $\mathbb{C} \setminus \{\beta\}$ (for some $\alpha, \beta \in \mathbb{C}$)

(ii) $z \mapsto \frac{1}{f(z)}$ yields an analytic isomorphism $U(\alpha) \to U(0)$,[22] and

(iii) $w \mapsto f(\frac{1}{w})$ yields an analytic isomorphism $U(0) \to U(\beta)$,

then there exists a composition inverse f^{-1} with the same properties. The automorphisms of $\hat{\mathbb{C}}$ consist of these maps f, together with the automorphisms of $\mathbb{C}$ (whose extensions to $\hat{\mathbb{C}}$ send $\infty \mapsto \infty$), and in fact we have

$$\text{(I.G.6)} \qquad \mathrm{Aut}(\hat{\mathbb{C}}) \cong \underline{\text{FLT}}.$$

[22] Here $U(a)$ denotes some neighborhood of a.

[*Heuristic sketch of* (I.G.6): That $\underline{\mathrm{FLT}} \hookrightarrow \mathrm{Aut}(\hat{\mathbb{C}})$ is easy: only the identity function yields the identity map. Now let f be an arbitrary complex-analytic automorphism of $\hat{\mathbb{C}}$. Then it is 1-to-1, hence has no essential singularities or higher-order poles. Since some point has to be sent to ∞, it has exactly one simple pole; similarly, there is one zero and it is also simple. If the pole and zero are at finite points α and $\beta \in \mathbb{C}$, then multiplying by $(z-\alpha)/(z-\beta)$ eliminates the zero and pole without introducing new ones. If the pole [resp. zero] is at ∞, then multiplying by $1/(z-\beta)$ [resp. $z-\alpha$] achieves the same effect. Either way, this leaves us with an analytic map from $\hat{\mathbb{C}}$ to $\mathbb{C}$. Since $\hat{\mathbb{C}}$ is compact, this is bounded, hence by Liouville constant. From how we arrived at this constant, we see that f is a quotient of two polynomials of degree ≤ 1, i.e. $f \in \underline{\mathrm{FLT}}$.]

This sketch (or unofficial proof) used things we don't yet know, including the Casorati–Weierstrass theorem and Liouville's theorem. Consider it motivation for the next part of the course!

So we have

$$\mathrm{SL}_2(\mathbb{C})/\{\pm\mathrm{id}\} \cong \underline{\mathrm{FLT}} \cong \mathrm{Aut}(\hat{\mathbb{C}}), \tag{I.G.7}$$

where the second isomorphism is unofficial. The group structure on $\underline{\mathrm{FLT}}$ clarifies some things:

- Composition inverses: using $ad - bc = 1$, we have
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
and thus
$$f^{-1}_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}(w) = \frac{dw-b}{-cw+a}.$$
Indeed, that $\underline{\mathrm{FLT}}$ is a group (under composition) means that all FLT's are 1-to-1.

- Iwasawa decomposition: for real FLT's (more on these below),
$$\mathrm{SL}_2(\mathbb{R}) \cong N \cdot A \cdot K$$
with $N \cong \mathbb{R}$, $A \cong \mathbb{R}_{>0}$, and $K \cong S^1$, by writing any $M \in \mathrm{SL}_2(\mathbb{R})$ *uniquely* as
$$M = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$
This decomposition has vast generalizations in the theory of linear algebraic groups. Here it means we can write an arbitrary FLT as a composition of three much simpler ones.

- Why 2×2 matrices? If you think of $\hat{\mathbb{C}}$ as
$$\mathbb{CP}^1 = \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{(z_1, z_2) \underset{\alpha \in \mathbb{C}^*}{\sim} (\alpha z_1, \alpha z_2)} \overset{\cong}{\rightleftarrows} \hat{\mathbb{C}}$$
$$\begin{aligned} [z:1] &\longleftarrow\!\shortmid\, z \neq \infty \\ (z_2 \neq 0)\ [z_1 : z_2] &\longmapsto \frac{z_1}{z_2} \\ [1:0] &\longleftarrow\!\shortmid\ \infty \\ [z_1 : 0] &\longmapsto \infty \end{aligned}$$
and write elements $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ instead of $[z_1 : z_2]$, then
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}$$
in $\mathbb{CP}^1$ becomes
$$z \left(= \frac{z_1}{z_2} \right) \mapsto \frac{az_1 + bz_2}{cz_1 + dz_2} = \frac{a(z_1/z_2) + b}{c(z_1/z_2) + d} = \frac{az + b}{cz + d}.$$
This reveals the reason both for the terminology and for the naturality of the matrix formulation: fractional linear transformations are *linear transformations* acting on lines in $\mathbb{C}^2$ through the origin.

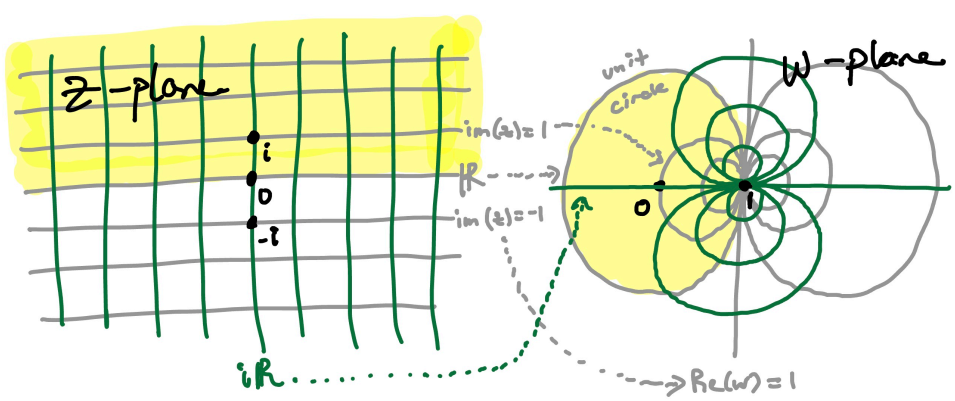
§ Action on circles.

Let $\mathscr{C}$ denote the set of circles and lines in $\mathbb{C}$. Recall from Chapter I.A that if we view $\hat{\mathbb{C}}$ as a sphere via stereographic projection, $\mathscr{C}$ is identified with the set of circles on $\hat{\mathbb{C}}$.

I.G.8. THEOREM. *(a) Any* $f \in \underline{\mathrm{FLT}}$ *takes* $\mathscr{C} \to \mathscr{C}$.
(b) This defines a transitive group action of $\underline{\mathrm{FLT}}$ *on* $\mathscr{C}$.

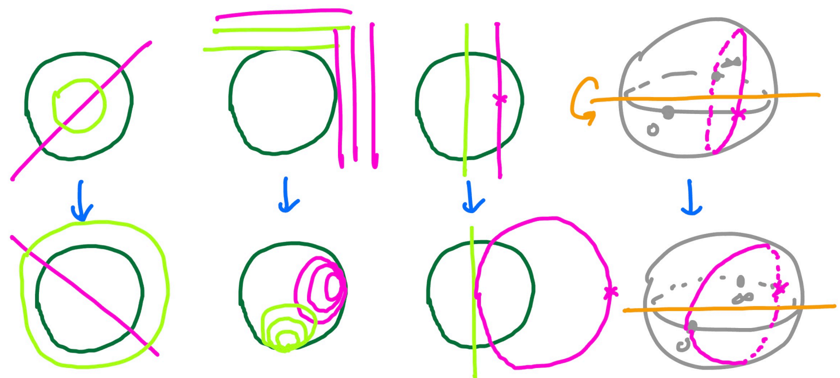
I.G.9. EXAMPLES. (1) Lines are special kinds of circles — the ones that contain ∞.

(2) The map $z \mapsto \frac{z-\mathbf{i}}{z+\mathbf{i}} = w$ takes $\mathfrak{H} \xrightarrow{\cong} D_1$, and has the following effect on horizontal and vertical lines:



In some sense, the last picture ("degenerate Steiner circles") is telling you what Cartesian coordinates look like at ∞ (about $w = 1$). The intersecting circles meet at right angles due to the "conformality" of FLT's (see Chapter I.H).

(3) Inversion: $z \mapsto \frac{1}{z} =: J(z)$ [unit circle shown in dark green]



The action of J on the spherical model of $\hat{\mathbb{C}}$ looks like flipping the sphere $180°$ about the orange line (see rightmost picture).

(4) $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ sends $\mathbb{R} \to \mathbb{R}$, and these are the only FLT's which preserve the reals. If in addition $ad - bc > 0$, then f also sends $\mathfrak{H} \to \mathfrak{H}$. These are automorphisms of $\hat{\mathbb{C}}$ which (in the spherical picture) don't mix the upper and lower hemispheres.

The $N \cdot A \cdot K$ decomposition of $\mathrm{SL}_2(\mathbb{R})$ certainly suggests that these FLT's preserve circles: translation by x; dilation by y; the only issue is what the "K" matrices/FLT's do. In the proof that follows (for *all* FLT's), we'll make use of a different decomposition.

PROOF OF THEOREM I.G.8(a). Let $F(z) = \frac{az+b}{cz+d}$ be given, with $a, b, c, d \in \mathbb{C}$. First suppose that $c = 0$. Then

$$F(z) = \tfrac{a}{d}z + \tfrac{b}{d} = \tau_{b/d} \circ \mu_{a/d},$$

where τ denotes translation by an element in $\mathbb{C}$, and μ denotes multiplication by element in $\mathbb{C}^*$. Both operations preserve lines *and* circles.

If $c \neq 0$, then

$$F(z) = \frac{az + \frac{d}{c}a}{cz+d} + \frac{b - \frac{d}{c}a}{cz+d} = \frac{a}{c}\frac{\cancel{cz+d}}{\cancel{cz+d}} + \frac{\frac{b}{c} - \frac{da}{c^2}}{z + \frac{d}{c}}$$
$$= \tau_{a/c} \circ \mu_{\frac{bc-da}{c^2}} \circ J \circ \tau_{d/c}.$$

We know that τ and μ preserve $\mathscr{C}$; how about J? If it does, then F preserves $\mathscr{C}$ and we're done.

Writing $J(x+\mathbf{i}y)=u+\mathbf{i}v$, since J^2 is the identity map we get

$$x+\mathbf{i}y = J(u+\mathbf{i}v) = \frac{1}{u+\mathbf{i}v} = \frac{u}{u^2+v^2} + \mathbf{i}\frac{-v}{u^2+v^2}.$$

Hence, given an object

$$\mathfrak{C} = \{A(x^2+y^2)+Bx+Cy+D=0\}$$

of $\mathscr{C}$, we can rewrite the equation in u,v to get

$$\begin{aligned} J(\mathfrak{C}) &= \left\{A\left(\frac{u^2+v^2}{(u^2+v^2)^2}\right)+B\frac{u}{u^2+v^2}-C\frac{v}{u^2+v^2}+D=0\right\} \\ &= \{A+Bu-Cv+D(u^2+v^2)=0\} \end{aligned}$$

which is again in $\mathscr{C}$. □

To prove part (b) of the Theorem, which is to say transitivity of the action, we'll need some preliminary results:

I.G.10. LEMMA. *Given z_1,z_2,z_3 distinct in $\hat{\mathbb{C}}$, and w_1,w_2,w_3 distinct in $\hat{\mathbb{C}}$, there exists a unique $f_M \in \underline{\mathrm{FLT}}$ sending $z_i \mapsto w_i$ $(i=1,2,3)$.*

PROOF OF EXISTENCE. First, we write down FLT's f and g sending z_1,z_2,z_3 [resp. w_1,w_2,w_3] to $0,1,\infty$:

$$f(z)=\frac{z-z_1}{z-z_3}\cdot\frac{z_2-z_3}{z_2-z_1}, \quad g(w)=\frac{w-w_1}{w-w_3}\cdot\frac{w_2-w_3}{w_2-w_1}.$$

It follows that $(g^{-1}\circ f)(z_i)=w_i$ $(\forall i)$. So take $w=g^{-1}(f(z))$, i.e. $f(z)=g(w)$. □

I.G.11. EXAMPLE. To find f_M sending

$$z_1,z_2,z_3=-1,0,1 \quad \longmapsto \quad -1,\mathbf{i},1=w_1,w_2,w_3,$$

set $f(z)=g(w)$ as in the above proof:

$$\begin{gathered} \frac{z+1}{z-1}\cdot\frac{-1}{1} = \frac{w+1}{w-1}\cdot\frac{\mathbf{i}-1}{\mathbf{i}+1} \\ (z+1)(w-1) = -\mathbf{i}(w+1)(z-1) \\ zw+w-z-1 = -\mathbf{i}wz-\mathbf{i}z+\mathbf{i}w+\mathbf{i} \end{gathered}$$

$$\implies w = \tfrac{z+\mathbf{i}}{\mathbf{i}z+1} = f_M(z) \implies M = \begin{pmatrix} 1 & \mathbf{i} \\ \mathbf{i} & 1 \end{pmatrix}.$$

PROOF OF UNIQUENESS. If f, g both map $z_i \mapsto w_i$ $(\forall i)$, then the composition $g^{-1} \circ f$ fixes the $\{z_i\}$. Let F send $\{z_i\}$ to $\{0, 1, \infty\}$. Then $h = F \circ g^{-1} \circ f \circ F^{-1}$ fixes $0, 1, \infty$

$$\begin{aligned}
\implies \quad h(z) = \frac{az+b}{cz+d} &: \quad \infty \mapsto \infty \quad \implies \quad c = 0 \ (d, a \neq 0) \\
\implies \quad = Az + B &: \quad 0 \mapsto 0 \quad \implies \quad B = 0 \\
\implies \quad = Az &: \quad 1 \mapsto 1 \quad \implies \quad A = 1 \\
\implies \quad = z = \mathrm{id}(z). &
\end{aligned}$$

So $\mathrm{id} = F \circ g^{-1} \circ f \circ F^{-1} \implies F^{-1} \circ \mathrm{id} \circ F = g^{-1} \circ f \implies \mathrm{id} = g^{-1} \circ f \implies g = f$. $\square$

I.G.12. COROLLARY. *There exists a unique $\mathfrak{C} \in \mathscr{C}$ through any three distinct $z_i \in \hat{\mathbb{C}}$.*

PROOF. (Existence) The Lemma provides $f \in \underline{\mathrm{FLT}}$ sending

$$0, 1, \infty \mapsto z_1, z_2, z_3.$$

Since $\hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ belongs to $\mathscr{C}$, part (a) of the Theorem implies $f(\hat{\mathbb{R}}) \in \mathscr{C}$; moreover, $f(\hat{\mathbb{R}})$ contains the $\{z_i\}$. So take $\mathfrak{C} = f(\hat{\mathbb{R}})$.

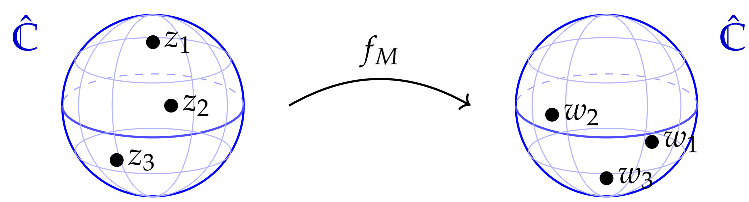
(Uniqueness) If there are two, then applying f^{-1} gives 2 elements of $\mathscr{C}$ through $0, 1, \infty$. But there is no circle in $\mathbb{C}$ through these points, and the only line is $\mathbb{R}$. $\square$

I.G.13. REMARK. Given $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ distinct, we write $\mathfrak{C}_{(z_1,z_2,z_3)}$ for the circle (or line) guaranteed by Corollary I.G.12.

PROOF OF THEOREM I.G.8(b). Given $\mathfrak{C}, \mathfrak{C}' \in \mathscr{C}$, take $z_1, z_2, z_3 \in \mathfrak{C}$ distinct, $w_1, w_2, w_3 \in \mathfrak{C}'$ distinct, and let f send $z_i \mapsto w_i$ $(\forall i)$. Then $f(\mathfrak{C}) \in \mathscr{C}$ and contains the $\{w_i\}$. By the Corollary, $f(\mathfrak{C}) = \mathfrak{C}'$. $\square$

§ **Cross ratios.**

Let z_1, z_2, z_3 and w_1, w_2, w_3 be three triples of distinct points in $\hat{\mathbb{C}}$. By Lemma I.G.10, there exists a unique $f_M \in \underline{\text{FLT}}$ taking $z_i \mapsto w_i$ $(\forall i)$.



In the special case where $\{w_i\}$ are $1, 0, \infty$, this is

$$f(z) = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} =: \mathrm{CR}(z, z_1, z_2, z_3) \tag{I.G.14}$$

which tells us where z is sent by the FLT taking the last three entries to $1, 0, \infty$ (in that order).[23]

I.G.15. THEOREM. (a) CR *is invariant under fractional linear transformations: if* $g \in \underline{\text{FLT}}$, *then*

$$\mathrm{CR}(g(z), g(z_1), g(z_2), g(z_3)) = \mathrm{CR}(z, z_1, z_2, z_3). \tag{I.G.16}$$

(b) z, z_1, z_2, z_3 *belong to the same* $\mathfrak{C} \in \mathscr{C}$ $\iff$ $\mathrm{CR}(z, z_1, z_2, z_3) \in \hat{\mathbb{R}}$.

PROOF. (a) $f \circ g^{-1}$ sends

$$g(z_1), g(z_2), g(z_3) \longmapsto 1, 0, \infty.$$

So LHS(I.G.16) $= (f \circ g^{-1})(g(z)) = f(z) =$ RHS(I.G.16).

(b) ($\implies$): If z lies on $\mathfrak{C} := \mathfrak{C}_{(z_1, z_2, z_3)}$ then (by transitivity of action of $\underline{\text{FLT}}$ on $\mathscr{C}$) there is some $F \in \underline{\text{FLT}}$ taking $\mathfrak{C}$ to $\hat{\mathbb{R}}$. But then $\mathrm{CR}(z, z_1, z_2, z_3) = \mathrm{CR}(F(z), F(z_1), F(z_2), F(z_3)) \in \hat{\mathbb{R}}$.

($\impliedby$): Let $G \in \underline{\text{FLT}}$ send $1, 0, \infty$ to z_1, z_2, z_3 (hence $\hat{\mathbb{R}}$ to $\mathfrak{C} := \mathfrak{C}_{(z_1, z_2, z_3)}$). Then

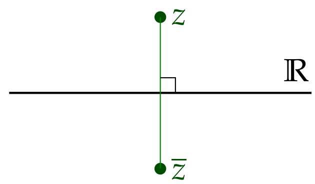
$$G^{-1}(z) = \mathrm{CR}(G^{-1}(z), 1, 0, \infty) = \mathrm{CR}(z, z_1, z_2, z_3) \in \hat{\mathbb{R}}$$

yields $z = G(G^{-1}(z)) \in G(\hat{\mathbb{R}}) = \mathfrak{C}$. $\square$

[23] Different variants exist in the literature; some authors use $0, 1, \infty$.

§ Symmetry and orientation.

Complex conjugation is a reflection about $\mathbb{R}$, with the "perpendicularity" property suggested by the picture:



Are there corresponding reflections about arbitrary $\mathfrak{C} \in \mathscr{C}$? Let $F \in \underline{\mathrm{FLT}}$ take $\hat{\mathbb{R}} \to \mathfrak{C}$, and define

$$z^*_{\mathfrak{C}} := F(\overline{F^{-1}(z)}) = (F \circ \overline{F}^{-1})(\bar{z}). \tag{I.G.17}$$

(The bar denotes complex conjugation, and $\overline{F}$ means $F_{\overline{M}}$ if $F = F_M$.) Given any other $\widetilde{F} \in \underline{\mathrm{FLT}}$ taking $\hat{\mathbb{R}} \to \mathfrak{C}$,

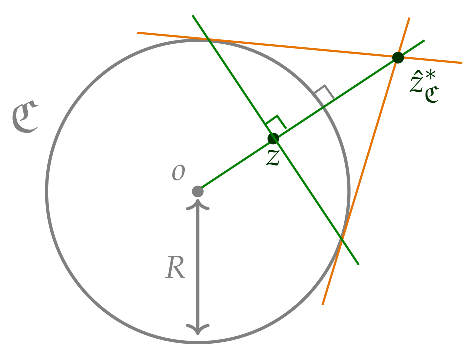
$$F^{-1} \circ \widetilde{F} = f_M, \qquad M \in \mathrm{SL}_2(\mathbb{R}).$$

So $\overline{f_M} = f_M$ and

$$\begin{aligned} \widetilde{F} = F \circ f_M \implies \widetilde{F}(\overline{\widetilde{F}^{-1}(z)}) &= F(f_M(\overline{f_M^{-1}(F^{-1}(z))})) \\ &= F(f_M(f_M^{-1}(\overline{F^{-1}(z)}))) = F(\overline{F^{-1}(z)}) \end{aligned}$$

and $z^*_{\mathfrak{C}}$ is well-defined.

The above picture holds for any line, since we can take F to be a composite of rotations, dilations and translations. For any line or circle, $(\cdot)^*_{\mathfrak{C}}$ fixes points on $\mathfrak{C}$ (why?). Now consider the geometric construction of "$\hat{z}^*_{\mathfrak{C}}$" from **[Ahlfors]** shown at right.



I.G.18. Proposition. $z^*_{\mathfrak{C}} = \hat{z}^*_{\mathfrak{C}}$.

Sketch. It suffices to check this for $\mathfrak{C} = \partial D_1 = \{|z| = 1\}$, since we can translate the center o to the origin 0 and dilate. This operation "commutes" with the reflection operations $z \mapsto z^*_{\mathfrak{C}}$ and $z \mapsto \hat{z}^*_{\mathfrak{C}}$.

Begin by writing $z = re^{\mathbf{i}\theta}$. The FLT

$$z \mapsto \frac{z - \mathbf{i}}{z + \mathbf{i}} = F(z)$$

sends $\hat{\mathbb{R}} \to \partial D_1$ and has $F^{-1}(z) = \frac{\mathbf{i}(1+z)}{1-z}$, whence

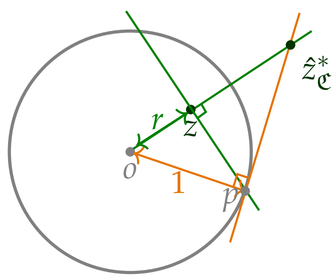
$$F(\overline{F^{-1}(z)}) = \frac{-\frac{\mathbf{i}(1+\bar{z})}{1-\bar{z}} - \mathbf{i}}{-\frac{\mathbf{i}(1+\bar{z})}{1-\bar{z}} + \mathbf{i}} = \cdots = \frac{1}{\bar{z}}.$$

That is,

$$(re^{\mathbf{i}\theta})^*_{\mathfrak{C}} = \frac{1}{re^{-\mathbf{i}\theta}} = \frac{1}{r}e^{\mathbf{i}\theta},$$

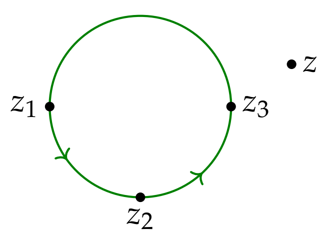
so $z^*_{\mathfrak{C}}$ lies on the line through o and z and $oz^*_{\mathfrak{C}}$ has length $\frac{1}{r}$.

Now consider the right triangles $op\hat{z}^*_{\mathfrak{C}}$ and ozp:



since these triangles are similar, the edge $o\hat{z}^*_{\mathfrak{C}}$ has length $\frac{1}{r}$, and so $\hat{z}^*_{\mathfrak{C}} = \frac{1}{r}e^{\mathbf{i}\theta} = z^*_{\mathfrak{C}}$ as promised. $\square$

The geometric picture now makes it clear that the segment connecting $z^*_{\mathfrak{C}}$ and z meets $\mathfrak{C}$ orthogonally. Moreover, they lie on different "sides" of $\mathfrak{C}$. If we use the ordering of z_1, z_2, z_3 on $\mathfrak{C}$ to give it an orientation,



then we say that z is to the **right** [resp. **left**] of the (oriented) circle if $\mathrm{Im}(\mathrm{CR}(z, z_1, z_2, z_3)) > 0$ [resp. < 0]. The circle is oriented **counterclockwise** if ∞ is to the right.

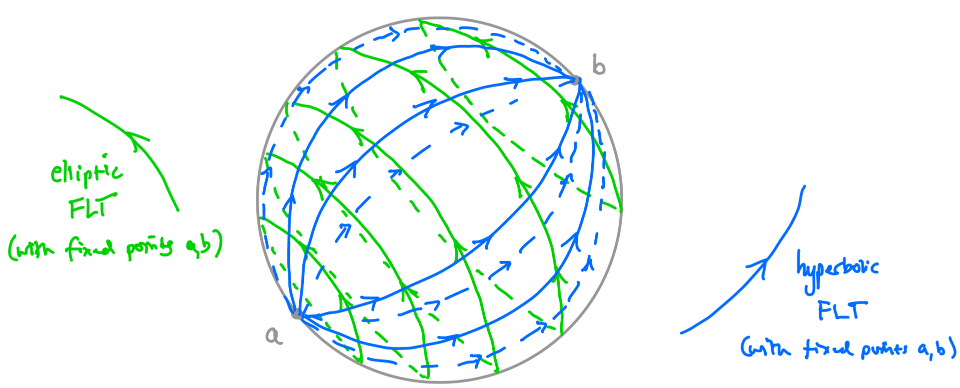
§ Fixed points.

Let $f(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. A **fixed point** of f is a solution to the equation $z = f(z)$, or equivalently

$$\gamma z^2 + (\delta - \alpha)z - \beta = 0 \implies z = \frac{\alpha - \delta \pm \sqrt{(\delta-\alpha)^2 + 4\gamma\beta}}{2\gamma}.$$

- If $\gamma = 0$ then one or both fixed points (counted with multiplicity) are at ∞. In this case, f is "affine" or "loxodromic".[24]
- If $\gamma \neq 0$ and $(\delta - \alpha)^2 = -4\gamma\beta$ then there is one (finite) fixed point with multiplicity 2, and f is "parabolic".
- If there are 2 distinct finite roots, then call them a and b, and take σ to be a FLT sending $a, b \mapsto 0, \infty$, like $\sigma(z) = \frac{z-a}{z-b}$. Then $\sigma \circ f \circ \sigma^{-1}$ fixes $0, \infty$. (Note that this simply diagonalizes the corresponding matrix.) The FLT's fixing 0 and ∞ are just the μ_ξ ($\xi \in \mathbb{C}^*$), i.e. the dilation-rotations. So $\sigma \circ f \circ \sigma^{-1} = \mu_\xi$. If $\xi \in \mathbb{R}^*$ then f is "hyperbolic"; if $\xi \in S^1$ "elliptic"; otherwise (again) "loxodromic".

I.G.19. REMARK. (a) The pictures of circles on [**Ahlfors**, p. 85] are just the image under $z \mapsto \frac{az+b}{z+1}$ (sends $0, \infty \mapsto b, a$) of polar coordinate lines. The meaning is clearer on $\hat{\mathbb{C}}$:



(b) The transformation

$$z \mapsto \frac{(\cos\theta)z + \sin\theta}{-(\sin\theta)z + \cos\theta}$$

[24]Terminology is that of [**Ahlfors**]; see the next page for a suggested revision and clarification.

fixes $\mathbf{i}$ and $-\mathbf{i}$. Diagonalizing the corresponding matrix, or just computing the characteristic polynomial, you find the eigenvalues are $\cos(\theta) + \mathbf{i}\sin(\theta)$, i.e. $|\xi| = 1 \implies$ elliptic. (In general, ξ is the ratio of the eigenvalues.)

In the factors of the Iwasawa decomposition,

- $N =$ affine FLT's (or parabolic if you count ∞ as a legitimate FP)
- $A =$ also affine (or hyperbolic if you count ∞ as a legitimate FP)
- $K =$ elliptic FLT's.

(c) In fact, I would propose the following (revised) terminology: working modulo $\pm$id, with $M \in \mathrm{SL}_2(\mathbb{C})$,

- f_M is **elliptic** $\iff$ M conjugate to $\begin{pmatrix} e^{\mathbf{i}\theta} & 0 \\ 0 & e^{-\mathbf{i}\theta} \end{pmatrix}$, $\theta \in \mathbb{R}$;
- f_M is **hyperbolic** $\iff$ M conjugate to $\begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$, $r \in \mathbb{R}_{>0}$; and
- f_M is **parabolic** $\iff$ M conjugate to $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\alpha \in \mathbb{C}$.

Then every FLT is a product of elliptic, hyperbolic, and parabolic transformations, due to the Jordan decomposition.

Exercises

(1) Find the fractional linear transformation which carries $0, \mathbf{i}, -\mathbf{i}$ into $1, -1, 0$.

(2) Express the cross ratios corresponding to the 24 permutations of four points in terms of $\lambda = \mathrm{CR}(z_1, z_2, z_3, z_4)$.

(3) Find all fractional linear transformations which represent rotations of the Riemann sphere.

(4) (i) Assuming $ad - bc = 1$, prove that $f(z) = \frac{az+b}{cz+d}$ and its inverse map $\mathfrak{H}$ (the upper half plane) into $\mathfrak{H}$ if and only if $a, b, c, d \in \mathbb{R}$.
(ii) Prove (in one line) that these maps are automorphisms of $\mathfrak{H}$.