I.H. Conformal mappings

§ Criteria for conformality.

Let's first nail down a definition. Let $U, V \subset \mathbb{C}$ be regions, and $F : U \to V$ a mapping with continuous partial derivatives.

I.H.1. DEFINITION. F is **conformal** on an open subset $S \subseteq U$
$\overset{\text{def}}{\Longleftrightarrow}$ $F|_S$ preserves angles to first order
$\overset{\substack{\text{linear}\\\text{algebra}}}{\Longleftrightarrow}$ $J_F(z)$ is nonzero of the form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ $(\forall z \in S)$
$\overset{\text{(I.B)}}{\Longleftrightarrow}$ $F|_S$ is holomorphic with nonvanishing (complex) derivative.

How do we know a holomorphic (or analytic) map f is conformal? Though it will turn out that it is enough for f to be 1-to-1, for now we content ourselves with the

I.H.2. PROPOSITION. *If f has a 2-sided holomorphic inverse $g : V \to U$, then f is conformal on all of U.*

PROOF. $1 = (g \circ f)'(z_0) = g'(f(z_0)) \cdot f'(z_0) \implies f'(z_0) \neq 0.$ □

In this situation, f is said to be a **biholomorphism** or **holomorphic isomorphism** $U \overset{\cong}{\to} V$, and also (because of the Proposition) a **conformal equivalence**.

When are two open sets in $\mathbb{C}$ conformally equivalent? Here is a famous result:

I.H.3. RIEMANN MAPPING THEOREM. *Given $U \subsetneq \mathbb{C}$ a simply connected region, there exists a conformal equivalence $f : U \overset{\cong}{\to} D_1$.*

I.H.4. EXAMPLE. Taking $U = \mathfrak{H}$ and $f(z) := \frac{z-\mathbf{i}}{z+\mathbf{i}}$, I claim that

$$\mathfrak{H} \xrightarrow[\cong]{f} D_1.$$

First regard f as a FLT; as such, it is an analytic automorphism of $\hat{\mathbb{C}}$. In particular, it is 1-to-1.

Now we check that $f(\mathfrak{H}) \subseteq D_1$: let $z \in \mathfrak{H}$, so $z = x + \mathbf{i}y, y > 0$. We proceed as follows:

$$\begin{aligned}
y > 0 &\implies y^2 - 2y + 1 < y^2 + 2y + 1 \\
&\implies (y-1)^2 < (y+1)^2 \\
&\implies x^2 + (y-1)^2 < x^2 + (y+1)^2 \\
&\implies |z - \mathbf{i}|^2 < |z + \mathbf{i}|^2 \\
&\implies |z - \mathbf{i}| < |z + \mathbf{i}| \\
&\implies |f(z)| < 1 \\
&\implies f(z) \in D_1.
\end{aligned}$$

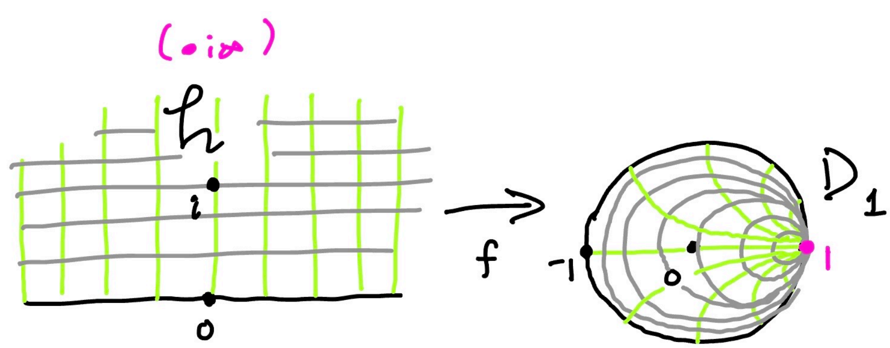
Finally, we must show that $f(\mathfrak{H}) \supseteq D_1$: for $w \in D_1$ we have

$$\begin{aligned}
\mathrm{Im}\left(-\mathbf{i}\frac{w+1}{w-1}\right) &= \mathrm{Re}\left(-\frac{w+1}{w-1}\right) \\
&= \mathrm{Re}\left(-\frac{(w+1)(\overline{w}-1)}{|w-1|^2}\right) \\
&= \mathrm{Re}\left(-\frac{|w|^2 - 1 + \overline{w} - w}{|w-1|^2}\right) \\
&= \frac{1-|w|^2}{|w-1|^2} > 0
\end{aligned}$$

so that $-\mathbf{i}\frac{w+1}{w-1} \in \mathfrak{H}$, and then

$$f\left(-\mathbf{i}\frac{w+1}{w-1}\right) = \frac{-\mathbf{i}\frac{w+1}{w-1} - \mathbf{i}}{-\mathbf{i}\frac{w+1}{w-1} + \mathbf{i}} = \frac{w+1+(w-1)}{w+1-(w-1)} = w$$

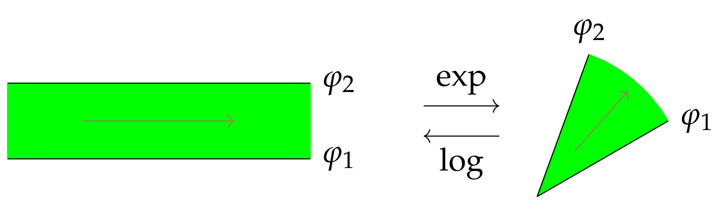
ensures $w \in f(\mathfrak{H})$.



§ Constructing conformal equivalencies.

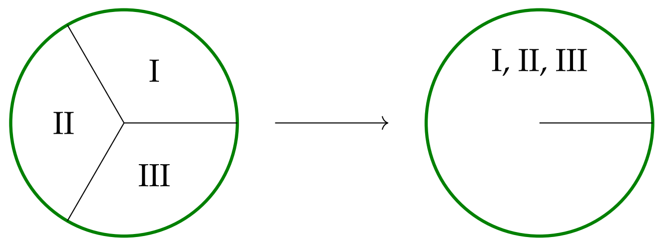
When we cover the proof of the Riemann Mapping Theorem in Complex Analysis II, we will develop more sophisticated methods to construct conformal equivalencies, simply by considering the map along the boundary. For now, the tools are for constructing or studying maps are:

- Fractional linear transformations. These give conformal equivalencies from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. It's also easy to "see" what they do since they take circles/lines to circles/lines.
- Powers. Taking $0 < \varphi_2 - \varphi_1 \leq 2\pi$, the map $z \mapsto z^\alpha = e^{\alpha \log z}$ takes
$$S(\varphi_1, \varphi_2) := \{z \in \mathbb{C}^* \mid \varphi_1 < \arg z < \varphi_2\}$$
to "$S(\alpha\varphi_1, \alpha\varphi_2)$", which makes sense expressed this way — and is 1-to-1 — iff $0 < \alpha(\varphi_2 - \varphi_1) \leq 2\pi$. (In particular, the map sends $e^{\mathbf{i}\varphi} \mapsto e^{\alpha \log(e^{\mathbf{i}\varphi})} = e^{\mathbf{i}\alpha\varphi}$.)
- Level curves. We know that
$$F(x = x_0) \textit{ and } F(y = y_0) \textit{ are orthogonal (in the image)}, \text{ and}$$
$$F^{-1}(u = u_0) \textit{ and } F^{-1}(v = v_0) \textit{ are orthogonal (in the domain).}$$
We can also apply this observation to polar coordinate "axes".
- Exp and log. The exponential map $z \mapsto e^z$ sends strips to sectors:
$$\{x + \mathbf{i}y \mid y \in (\varphi_1, \varphi_2)\} \underset{\text{1-to-1}}{\longmapsto} S(\varphi_1, \varphi_2)$$



The rest of this section consists of worked examples of conformal equivalencies.

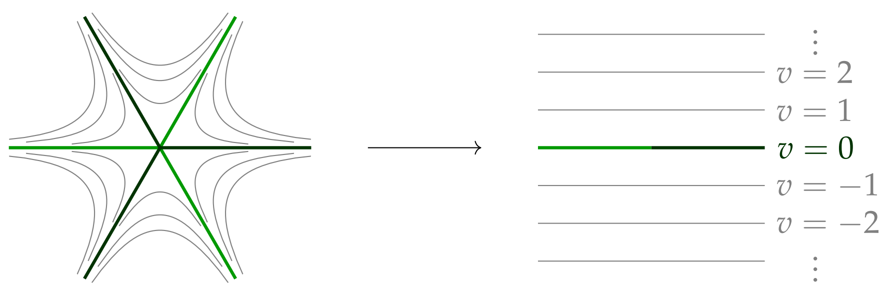
I.H.5. EXAMPLE. $z \mapsto z^3$ gives conformal equivalencies of the regions labeled "I, II, III" inside the unit disk with the slit unit disk:



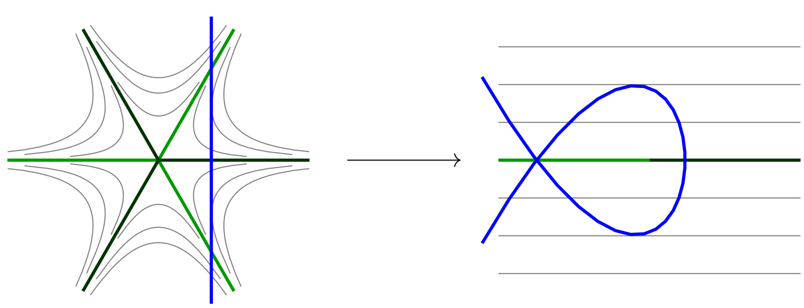
The preimage of $v = v_0$ is given by

$$v_0 = \operatorname{Im}((x + \mathbf{i}y)^3) = 3yx^2 - y^3 = y(3x^2 - y^2).$$

If $v_0 = 0$, this is $\frac{y}{x} = 0$, $\sqrt{3}$, or $-\sqrt{3}$.



Rotating the left-hand figure by 30° gives the preimage of the vertical lines. (The curves in the rotated figure are orthogonal to those in this one.) By drawing a (say) vertical line on top of the figure showing the level curves, you can "see" that $x = x_0 > 0$ gets sent to a nodal cubic curve as in the following figure.



I.H.6. EXAMPLE. We construct a conformal equivalence

$$F\colon \hat{\mathbb{C}} \setminus [-1,1] \xrightarrow{\cong} D_1$$

by composing several simpler maps

$$\begin{array}{ccc} U = \hat{\mathbb{C}} \setminus [-1,1] & & z \\ \cong \Big\downarrow \text{1-1 since FLT} & & \Big\downarrow \\ \mathbb{C} \setminus \mathbb{R}_{\geq 0} & & \frac{1+z}{1-z} \\ \cong \Big\downarrow \sqrt{\cdot} & & \Big\downarrow \\ \mathfrak{H} & & \sqrt{\frac{1+z}{1-z}} \\ \cong \Big\downarrow v \mapsto \frac{v-\mathbf{i}}{v+\mathbf{i}} & & \Big\downarrow \\ D_1 & & F(z) \end{array}$$

the last of which is from Example I.H.4. This gives

$$\begin{aligned} F(z) = \frac{\sqrt{1+z} - \mathbf{i}\sqrt{1-z}}{\sqrt{1+z} + \mathbf{i}\sqrt{1-z}} &= \frac{(\sqrt{1+z} - \mathbf{i}\sqrt{1-z})^2}{1+z+1-z} \\ &= \frac{1}{2}(1+z-(1-z) - 2\mathbf{i}\sqrt{1-z^2}) \\ &= z - \mathbf{i}\sqrt{1-z^2} \\ &= z - \sqrt{z^2-1}. \end{aligned}$$

Setting $w = F(z)$ we solve for the inverse:

$$z - w = \sqrt{z^2-1}$$

$$z^2 - 2zw + w^2 = z^2 - 1$$

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right) \tag{I.H.7}$$

and we can "pull back" equations under this inverse map to see where a given curve goes under F.

For instance, consider the set of (confocal) ellipses in the z-plane with foci ± 1:

$$|z+1| + |z-1| = C \ (> 2)$$

Squaring both sides (and remembering that $|\alpha|^2 = \alpha\overline{\alpha}$), we have

$$2|z|^2 + 2|z^2 - 1| = C^2 - 2,$$

whereupon plugging in (I.H.7) yields

$$\frac{1}{2}\left(w + \frac{1}{w}\right)\left(\overline{w} + \frac{1}{\overline{w}}\right) + \frac{1}{2}\left|w^2 + w^{-2} - 2\right| = C^2 - 2;$$

and rewriting $|w^2 + w^{-2} - 2| = |w - w^{-1}|^2 = (w - w^{-1})(\overline{w} - \overline{w}^{-1})$,

$$|w|^2 + \frac{1}{|w|^2} = C^2 - 2$$

$$|w|^4 + (2 - C^2)|w|^2 + 1 = 0$$

$$\implies \quad |w|^2 = \frac{(C^2 - 2) \pm \sqrt{C^4 - 4C^2}}{2}.$$

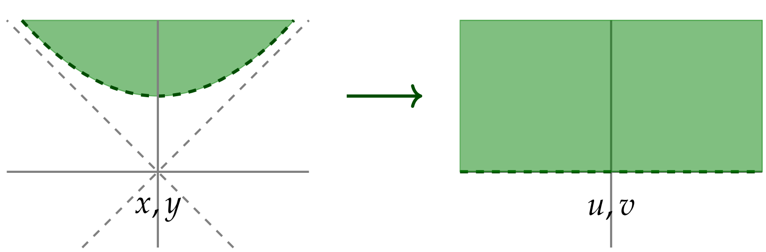
Choosing the minus sign to make $|w| < 1$,

$$|w|^2 = \frac{C^2 - 2 - C^2\sqrt{1 - \frac{4}{C^2}}}{2}$$

$$\implies \quad |w| = \sqrt{\frac{C^2}{2}\left(1 - \sqrt{1 - \frac{4}{C^2}}\right) - 1} =: R(C).$$

This shows that our ellipses map to circles of radius $R(C)$! For C large, $R(C) \approx \frac{1}{C^2}$; while for C "small" (i.e. $C \to 2^+$), $R(C) \to 1^-$ approaches the boundary of the unit disk.

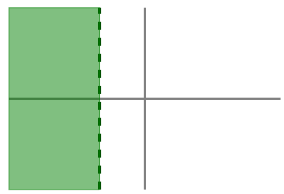
I.H.8. EXAMPLE. Let $\mathcal{R} = \{x + \mathbf{i}y \mid y^2 > x^2 + 1\}$ be the region above a hyperbola. To get started with a conformal map $\mathcal{R} \xrightarrow{\cong} \mathfrak{H}$



one might reason that $z \mapsto z^2$ would at least open up the $\frac{1}{4}$-plane $\mathcal{R}$ sits in, to a $\frac{1}{2}$-plane. Maybe that's some improvement? Actually, it's a *huge* improvement:

$$z^2 = (z + \mathbf{i}y)^2 = (x^2 - y^2) + \mathbf{i}(2xy) = u + \mathbf{i}v$$

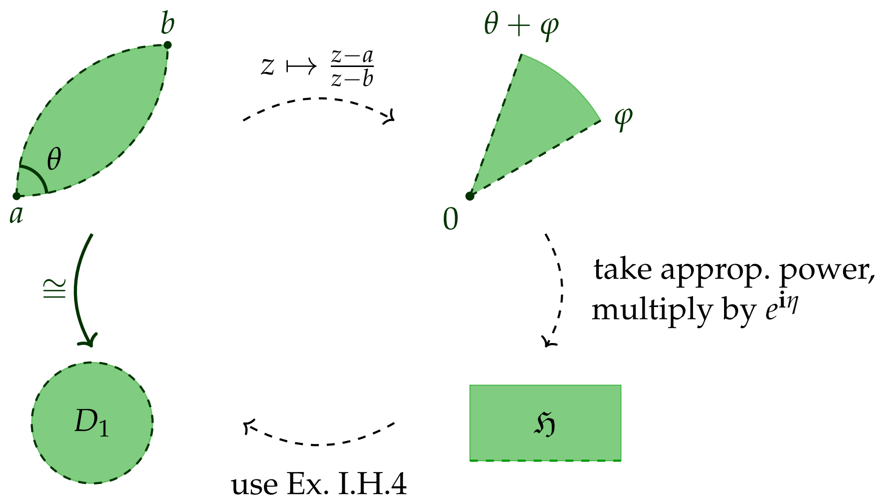
means that $x^2 - y^2 = -k$ (i.e. $y^2 = x^2 + k$) is sent by the squaring map to $u = -k$. Since $\mathcal{R}$ is filled out by hyperbolas $y^2 = x^2 + k$ for $k > 1$ (by definition) and these are mapped in 1-to-1 fashion onto lines $u = -k < -1$, this map sends $\mathcal{R}$ to the half-plane $\mathrm{Re}(z) < -1$.



All we then have to do is add 1 and multiply by $-\mathbf{i}$ to turn this into $\mathfrak{H}$: so the conformal equivalence is induced by

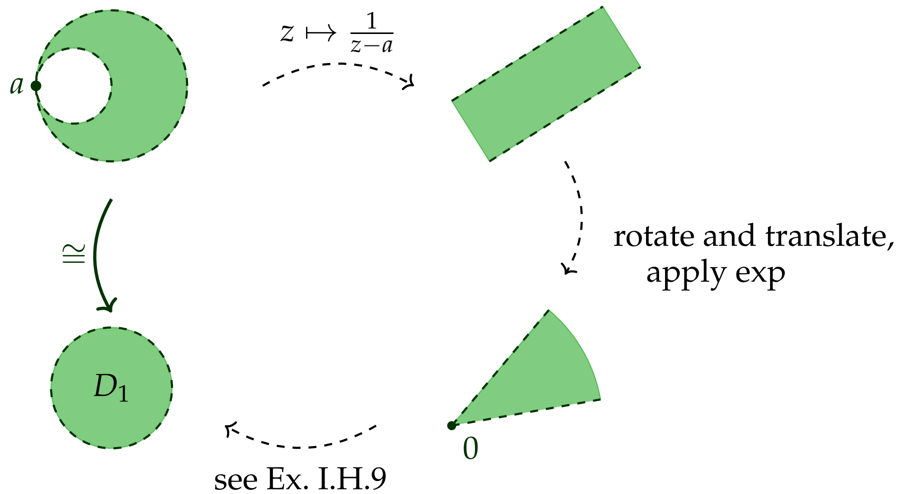
$$z \mapsto -\mathbf{i}(z^2 + 1).$$

I.H.9. EXAMPLE. Consider next the region enclosed by 2 circular arcs at upper left in the figure below. In order to map it conformally to the unit disk, we first apply a FLT to move a, b to $0, \infty$.

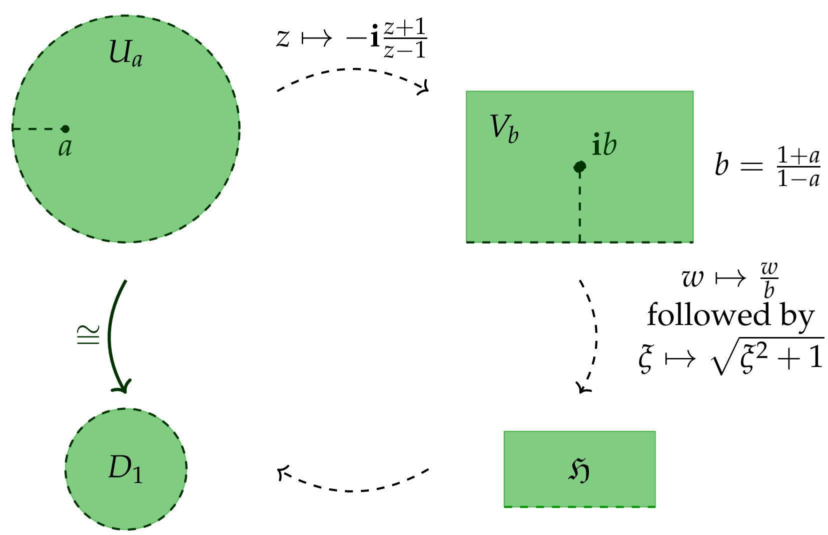


Since "circles through ∞" are lines, this maps our region to an angular wedge $S(\varphi, \theta + \varphi)$; since FLTs are conformal it has the same angle θ at the center. The rest is now straightforward.

I.H.10. EXAMPLE. Similarly, the region between two circles tangent at the origin can be mapped to the unit disk by a composition of familiar transformations.



I.H.11. EXAMPLE. Here we have a unit disk centered at 0, from which (for some $a \in (-1,1)$) the segment $(-1,a]$ has been removed. Call this U_a. The first step in mapping U_a to D_1 is to apply the map f^{-1} from Ex. I.H.4, which sends U_a isomorphically to the complement V_b of a segment in $\mathfrak{H}$:



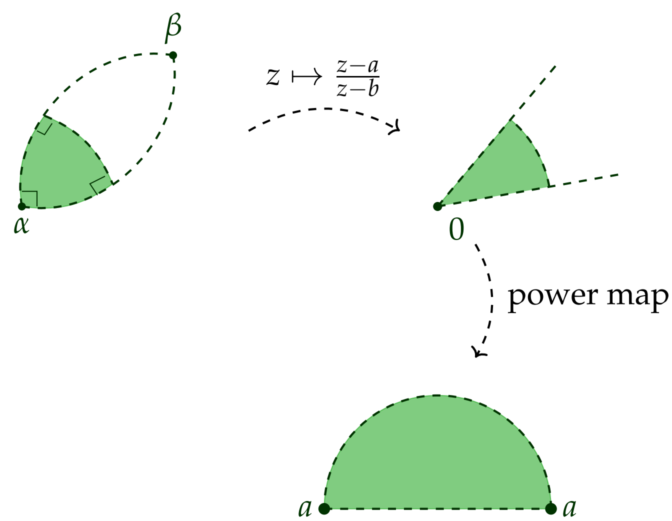
After rescaling b to 1, the squaring map sends V_1 to $\mathbb{C} \setminus \mathbb{R}_{\geq -1}$; adding 1 shifts this to $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$; square root sends this to $\mathfrak{H}$; and finally we map that to D_1. The full composition is

$$z \mapsto \frac{\sqrt{(-\frac{\mathbf{i}}{b}\frac{z+1}{z-1})^2+1}-\mathbf{i}}{\sqrt{(-\frac{\mathbf{i}}{b}\frac{z+1}{z-1})^2+1}+\mathbf{i}}.$$

It gives a conformal equivalence $\rho_a \colon U_a \overset{\cong}{\to} D_1$. Is it beautiful, or disgusting? You decide.

Notice that composing $(\rho_{a_1})^{-1} \circ \rho_{a_2}$ for two different values of a shows that all the U_a (for any $a \in (-1,1)$) are conformally equivalent. Maybe this is starting to convince you that despite the "rigidity" of holomorphic maps, they are flexible enough that the Riemann mapping theorem is plausible?

I.H.12. EXAMPLE. What if you have a curvy triangle with three right angles formed by circular arcs?

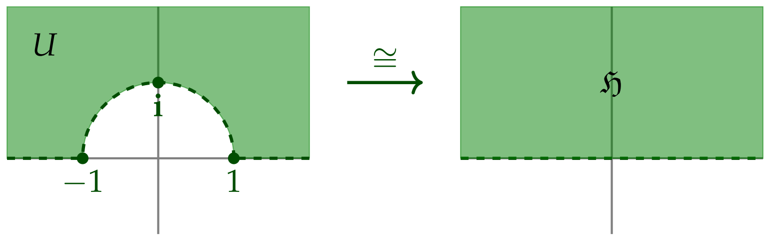


Well then, applying a FLT converts the region to a pie-shaped one with vertex at 0. Taking a real power puts us back in the situation of Example I.H.9, and from there we can get to the disk.

I.H.13. EXAMPLE. I claim that

$$f(z) = z + \frac{1}{z}$$

maps the region U conformally to the upper half-plane:



Begin by writing $z = x + \mathbf{i}y$ and

$$w = f(z) = x + \mathbf{i}y + \frac{x - \mathbf{i}y}{(x+\mathbf{i}y)(x-\mathbf{i}y)}$$
$$= x\left(1 + \frac{1}{x^2+y^2}\right) + \mathbf{i}y\left(1 - \frac{1}{x^2+y^2}\right).$$

We first check that $f(U) \subseteq \mathfrak{H}$:

$$\left\{ \begin{array}{c} |z| > 1 \\ \mathrm{Im}(z) > 0 \end{array} \right\} \implies \left\{ \begin{array}{c} x^2+y^2 > 1 \\ y > 0 \end{array} \right\} \implies \left\{ \begin{array}{c} \frac{1}{x^2+y^2} < 1 \\ y > 0 \end{array} \right\}$$
$$\implies \mathrm{Im}(w) = y \cdot \left(1 - \frac{1}{x^2+y^2}\right) > 0.$$

Next we show that $f(U) \supseteq \mathfrak{H}$. Given $w \in \mathfrak{H}$, we must solve $w = z + \frac{1}{z}$ for z in U:

$$wz = z^2 + 1$$
$$0 = z^2 - wz + 1 = (z - z_+)(z - z_-)$$

Clearly $z_+z_- = 1$. Note that $|z_+|$ and $|z_-|$ cannot be 1, because then

$$z_- = \overline{z_+} \implies w = z_+ + \overline{z_+} = 2\mathrm{Re}(z) \notin \mathfrak{H}$$

contradicts our choice of w. So let $|z_+| > 1$, $|z_-| < 1$; then

$$w = z_+ + \frac{1}{z_+} = z_+ + z_- \in \mathfrak{H} \implies \mathrm{Im}(z_+) > 0,$$

which together with $|z_+| > 1$ gives $z_+ \in U$.

Finally, we remark that $|z_-| < 1 \implies z_- \notin U$. So each $w \in \mathfrak{H}$ has a *unique* preimage under f, and f is 1-to-1. That the derivative is nowhere zero on U is left to you.

Exercises

(1) (a) Map the region between the circles $|z| = 1$ and $|z - \frac{1}{2}| = \frac{1}{2}$ onto a half plane.
(b) Find all circles (or lines) orthogonal to both of them.

(2) Conformally map the outside of the ellipse $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$ onto the open unit disk D_1.[25]

(3) Let $F(z) = \frac{z-\mathbf{i}}{z+\mathbf{i}}$. Find the image under F of:
(a) the upper half line $\mathbf{i}t$, with $t \geq 0$;
(b) the circle of center 1 and radius 1;
(c) the horizontal line $\mathbf{i} + t$, with $t \in \mathbb{R}$;
(d) the half circle $|z| = 2$ with $\mathrm{Im}(z) \geq 0$;
(e) the vertical line $\mathrm{Re}(z) = 1$ and $\mathrm{Im}(z) \geq 0$.

(4) (a) Show that $z \mapsto \sin(z)$ gives a conformal equivalence of $U := \{x + \mathbf{i}y \mid y > 0,\ x \in (0, \frac{\pi}{2})\}$ with $S(0, \frac{\pi}{2})$. [Think of sin as the composition of $\zeta = e^{\mathbf{i}z}$ with $w = \frac{\zeta - \zeta^{-1}}{2\mathbf{i}}$, and find a way to use some of the work done above and/or in [**Ahlfors**].]
(b) Describe the image under this map of (the intersection with U of) the Cartesian coordinate lines.

[25] N.B. You need to consider the complement of the closed ellipse in $\hat{\mathbb{C}}$. If ∞ is not included, the set is not simply connected and you'll only be able to map it to D_1 minus a point.