II. Complex Integration

II.A. Contour integrals

§ <u>Paths</u>.

Let $S \subseteq \mathbb{C}$ be a subset. In this section we explain

A (parametrized) **path** in S is a continuous (class $\mathcal{C}^0$) map

$$\gamma : [a,b] \to S.$$

Context will dictate whether "γ" refers to the path or its image.

A **partition** Π of $[a,b]$ is a finite set $\{t_0, \ldots, t_n\} \subset \mathbb{R}$ with

$$a = t_0 < t_1 < \cdots < t_n = b;$$

set $|\Pi| := \sup_i |t_i - t_{i-1}|$. The path γ is **rectifiable** if its length

$$\text{(II.A.1)} \qquad L(\gamma) := \operatorname*{lub}_{\Pi} \sum_{i=1}^{n} |\underbrace{\gamma(t_i) - \gamma(t_{i-1})}_{=:(\Delta\gamma)_i}| \qquad \text{is finite.}$$

Equivalently, we require that

$$\text{(II.A.2)} \qquad \begin{cases} \operatorname*{lub}_{\Pi} \sum_{i=1}^{n} |\mathrm{Re}(\gamma(t_i)) - \mathrm{Re}(\gamma(t_{i-1}))| \\ \operatorname*{lub}_{\Pi} \sum_{i=1}^{n} |\mathrm{Im}(\gamma(t_i)) - \mathrm{Im}(\gamma(t_{i-1}))| \end{cases} \qquad \text{are both finite.}$$

Since γ is compact, one also has directly

$$\text{(II.A.1)} \iff \int_\gamma f\,|dz| := \lim_{|\Pi|\to 0} \sum_{i=1}^{n} f(\gamma(t_i))\,|(\Delta\gamma)_i|$$

$$\text{is defined for every } f \in \mathcal{C}^0(S),$$

$$\text{and (II.A.2)} \iff \int_\gamma f\,dz := \lim_{|\Pi|\to 0} \sum_{i=1}^{n} f(\gamma(t_i))\,(\Delta\gamma)_i$$

$$\text{is defined for every } f \in \mathcal{C}^0(S).$$

So we get two notions of integral for any continuous function f on any rectifiable path γ. We will mainly use $\int_\gamma f\,dz$, called a **contour integral** (or line integral); whereas $\int_\gamma f\,|dz|$ is the contour integral *with respect to arclength* and is mainly applied to $|f|$ (rather than f) in bounding arguments.

Next, a (parametrized) **curve** in $\mathbb{C}$ is a continuously differentiable (class $\mathcal{C}^1$) map $\gamma\colon [a,b] \to \mathbb{C}$. It is sometimes called *regular* if the derivative is nowhere vanishing. To define the derivative, we can either write

$$\gamma(t) = x(t) + \mathbf{i}y(t) \;\rightsquigarrow\; \gamma'(t) = x'(t) + \mathbf{i}y'(t)$$

or

$$\lim_{h\to 0} \frac{\gamma(t+h)-\gamma(t)}{h} =: \gamma'(t).$$

This satisfies chain rules for compositions of the form

$$[c,d] \xrightarrow{\psi} [a,b] \xrightarrow{\gamma} S \quad \text{and} \quad [a,b] \xrightarrow{\gamma} S \xrightarrow{f} \mathbb{C}.$$

For the second type of composition,[1]

- $f \in \mathcal{C}^1(S) \implies (f\circ\gamma)' = f_x x' + f_y y' = \frac{\partial f}{\partial z}\gamma' + \frac{\partial f}{\partial \bar{z}}\overline{\gamma}'$
- $f \in \mathscr{H}\!ol(S) \implies (f\circ\gamma)' = (f'\circ\gamma)\cdot\gamma'$.

A $\mathcal{C}^1$ **path** (in S) $\gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a collection of curves

$$\gamma_j\colon [a_j, b_j] \to \mathbb{C}$$

satisfying $\gamma_{j+1}(a_{j+1}) = \gamma_j(b_j)$ (for $j = 1, \ldots, n-1$). These are the objects over which we shall be interested in integrating. By compactness of each of these subintervals, γ' is bounded; and so

$$\gamma \text{ is a } \mathcal{C}^1 \text{ path} \quad\implies\quad \gamma \text{ is rectifiable.}$$

II.A.3. EXAMPLE. Suppose $\gamma, \tilde{\gamma}\colon [-\epsilon,\epsilon] \to D_1$ are curves with $\gamma(0) = 0 = \tilde{\gamma}(0)$ and $\gamma'(0), \tilde{\gamma}'(0) \neq 0$. Let a function $f \in \mathscr{H}\!ol(D_1)$

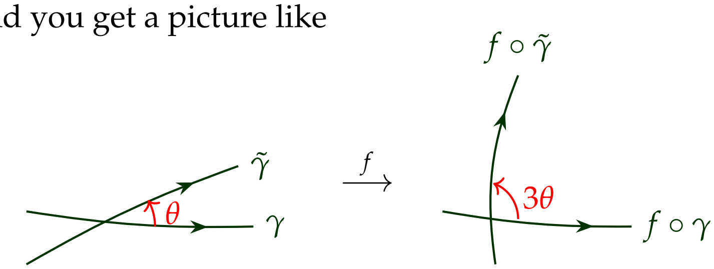
[1]Minor technical point: since we are not assuming $S \subseteq \mathbb{C}$ is open, $f \in \mathcal{C}^1(S)$ [resp. $\mathscr{H}\!ol(S)$] means that there is some open set $U \supseteq S$ to which f has a $\mathcal{C}^1$ [resp. holomorphic] extension.

with $f'(0) \neq 0$ be given. Then $f \circ \gamma$ and $f \circ \tilde{\gamma}$ are also curves and

$$\frac{(f \circ \tilde{\gamma})'(0)}{(f \circ \gamma)'(0)} = \frac{\cancel{f'(0)}\tilde{\gamma}'(0)}{\cancel{f'(0)}\gamma'(0)} = \frac{\tilde{\gamma}'(0)}{\gamma'(0)} \implies$$

$$\arg \tilde{\gamma}'(0) - \arg \gamma'(0) = \arg(f \circ \tilde{\gamma})'(0) - \arg(f \circ \gamma)'(0)$$

expresses conformality of f. On the other hand, if (say) $f(z) = z^3$ and $\gamma(t) = t$, $\tilde{\gamma}(t) = e^{\mathrm{i}\theta}t$, then $(f \circ \gamma)(u^{1/3}) = u$, $(f \circ \tilde{\gamma})(u^{1/3}) = e^{\mathrm{i}3\theta}u$ and you get a picture like



which reiterates the point that conformality is only true of holomorphic functions where their derivatives don't vanish.

Finally, we'll need some terminology related to global behavior of paths:

- γ is **closed** $\iff$ $\gamma(a) = \gamma(b)$; and
- γ is **simple** $\iff$ γ is 1-to-1.

II.A.4. JORDAN CURVE THEOREM. *If a path γ is simple and closed, then $\mathbb{C} \setminus \gamma$ has two connected components.*

§ Integration.

We now turn to a more convenient definition of the contour integral along a $\mathcal{C}^1$ path. Begin with a complex-valued $\mathcal{C}^0$ function on a closed interval

$$a = t_0 \quad t_1 \quad t_2 \quad \cdots \quad t_{N-1} \quad t_N = b \quad \xrightarrow[\text{(continuous)}]{G} \quad \mathbb{C}$$

written $G(t) = U(t) + \mathbf{i}V(t)$. Its integral

$$\int_a^b G\,dt := \int_a^b U\,dt + \mathbf{i}\int_a^b V\,dt$$

can be expressed as a limit

$$\lim_{N\to\infty} \frac{b-a}{N} \sum_{k=0}^{N-1} G(t_k)$$

where $t_k = a + k\frac{b-a}{N}$. We can bound it by

$$\begin{aligned} \left|\int_a^b G\,dt\right| &= \lim_{N\to\infty} \frac{b-a}{N} \left|\sum G(t_k)\right| \\ &\leq \lim_{N\to\infty} \frac{b-a}{N} \sum |G(t_k)| \\ &= \int_a^b |G| dt. \end{aligned}$$

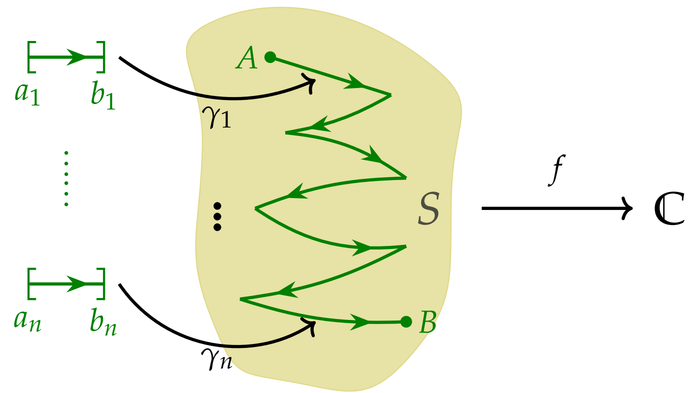
Things like integration by parts, the fundamental theorem of calculus, and (complex-)linearity hold as usual.

Now let $S \subseteq \mathbb{C}$ be a subset.

II.A.5. Definition. For any $\mathcal{C}^1$ path $\gamma = \sum \gamma_i$ in S and any function $f \in \mathcal{C}^0(S)$,

$$\int_\gamma f(z)dz \;:= \sum_i \int_{\gamma_i} f(z)\,dz \;:=\; \sum_i \int_{a_i}^{b_i} f(\gamma_i(t))\gamma_i'(t)\,dt.$$

The scenario is depicted on the next page.



II.A.6. REMARKS.

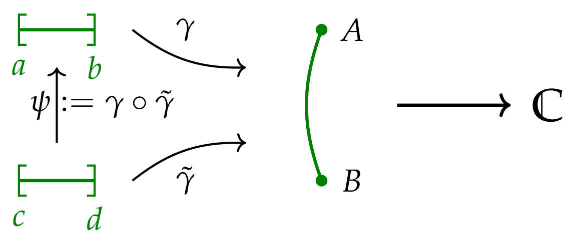
(i) This agrees with the earlier definition for rectifiable γ. (Why?)

(ii) $\int_\gamma (\alpha f + \beta g)\, dz = \alpha \int_\gamma f\, dz + \beta \int_\gamma g\, dz$.

(iii) "$-\gamma_i$" denotes the curve parametrized "with the opposite orientation" (i.e. backwards):

$$\int_{-\gamma_i} f\, dz = -\int_{\gamma_i} f\, dz.$$

(iv) Independence of parametrization: suppose two parametrizations of the same curve (with the same orientation) are given, viz.



Then the substitution $t = \psi(u)$ yields

$$\begin{aligned}\int_a^b f(\gamma(t))\gamma'(t)\, dt &= \int_c^d f(\gamma(\psi(u)))\gamma'(\psi(u))\psi'(u)\, du \\ &= \int_c^d f(\tilde{\gamma}(u))\tilde{\gamma}'(u)\, du.\end{aligned}$$

(v) Bounding integrals: put

$$L(\gamma) := \sum_i L(\gamma_i) = \sum_i \int_{a_i}^{b_i} |\gamma_i'(t)|dt = \int_\gamma |dz|,$$

where we note that $|\gamma_i'|$ is given by $\sqrt{(x_i')^2 + (y_i')^2}$; then

$$\begin{aligned} \left|\int_\gamma f(z)dz\right| &\leq \sum_i \left|\int_{a_i}^{b_i} f(\gamma_i(t))\gamma_i'(t)dt\right| \\ &\leq \sum_i \int_{a_i}^{b_i} \underbrace{|f(\gamma_i(t))|}_{\leq \|f\|_\gamma} |\gamma_i'(t)|dt \\ &\leq \|f\|_\gamma \sum_i L(\gamma_i) = \|f\|_\gamma L(\gamma) \qquad \text{(II.A.7)} \end{aligned}$$

(vi) Fundamental Theorem of Calculus (FTC): If $f \in C^0(S)$ has primitive $F \in \mathscr{Hol}(S)$ (i.e. $F' = f$) then

$$\begin{aligned} \int_\gamma f(z)dz &= \sum_i \int_{a_i}^{b_i} f(\gamma_i(t))\gamma_i'(t)dt = \sum_i \int_{a_i}^{b_i} (F \circ \gamma_i)'(t) \\ &= \sum_i (F(\gamma_i(b_i)) - F(\gamma_i(a_i))) \\ &= F(B) - F(A), \end{aligned}$$

where the last step is a collapsing sum.

(vii) Integration by parts: Now, if $g \in \mathcal{C}^1(S)$ (and $\gamma \subset S$, $f \in \mathcal{C}^0(S)$) there's nothing preventing us from defining $\int_\gamma f dg$. Here you need to think of

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = \frac{\partial g}{\partial z}dz + \frac{\partial g}{\partial \bar{z}}d\bar{z},$$

and

$$\begin{cases} dz = \gamma'(t)dt \\ d\bar{z} = \overline{\gamma'(t)}dt \end{cases}, \qquad \begin{cases} dx = \mathrm{Re}\gamma'(t)\,dt \\ dy = \mathrm{Im}\gamma'(t)\,dt \end{cases}.$$

This gives, assuming γ is a curve,

$$\begin{aligned}\int_\gamma f\,dg &= \int_a^b f(\gamma(t))\left\{\frac{\partial g}{\partial z}\gamma'(t)+\frac{\partial g}{\partial \bar{z}}\overline{\gamma'(t)}\right\}dt\\ &= \int_a^b f(\gamma(t))\left\langle(\nabla g)(\gamma(t)),\gamma'(t)\right\rangle dt,\end{aligned}$$

where the angle brackets denote dot product. Assuming *both* f and g are $\mathcal{C}^1$,

$$\int_\gamma g\,df = fg|_A^B - \int_\gamma f\,dg;$$

the proof is left to you.

II.A.8. EXAMPLE. We show how to compute

$$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\log(z)dz = 0 \tag{II.A.9}$$

via the bound (II.A.7), where $\log(z)$ denotes the branch of logarithm with $-\pi < \arg \leq \pi$. This has to be interpreted in the right way: $\int_{\partial D_\epsilon}$ is a limit of integrals over curves $\gamma\colon [-\pi+\delta,\pi-\delta]\to\mathbb{C}$ given by $t\mapsto \epsilon e^{\mathbf{i}t}$, as these avoid the branch cut. In any case the bound is

$$\begin{aligned}\left|\int_{\partial D_\epsilon}\log(z)dz\right| &\leq \|\log(z)\|_{\partial D_\epsilon}L(\partial D_\epsilon)\\ &\leq (|\log(\epsilon)|+\pi)2\pi\epsilon,\end{aligned}$$

and since $\epsilon\log(\epsilon)\to 0$ we get (II.A.9).

§ Loop integrals.

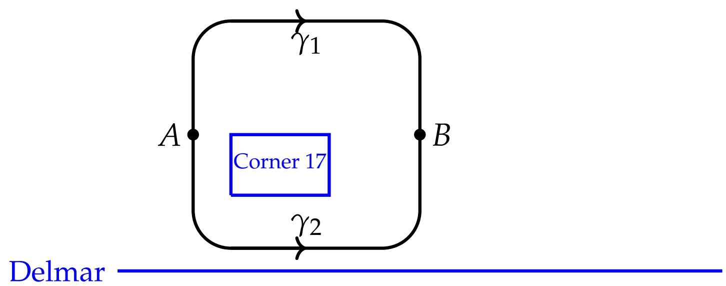
A **loop** is a closed path.

II.A.10. PROPOSITION. *Given $U \subset \mathbb{C}$ a region, and $f \in \mathcal{C}^0(U)$, the following are equivalent:*

(1) f has a holomorphic primitive on U (i.e. $\exists F \in Hol(U)$ s.t. $F' = f$);
(2) $\int_A^B f\,dz$ is independent of path ($\forall\ A, B \in U$); and
(3) $\int_\gamma f\,dz = 0$ ($\forall\ \mathcal{C}^1$ loops $\gamma \subset U$).

II.A.11. REMARK. It's sort of amazing that you only need to assume $f \in \mathcal{C}^0(U)$ to get (3) $\implies$ (1).

PROOF. (3) $\implies$ (2): Referring to the picture,



since $\gamma := \gamma_2 - \gamma_1$ is a loop, (3) gives

$$0 = \int_\gamma f\,dz = \int_{\gamma_2} f\,dz - \int_{\gamma_1} f\,dz.$$

(2) $\implies$ (3): Break a given loop into $\gamma_2 - \gamma_1$, use independence of path to get $\int_\gamma f\,dz$.

(1) $\implies$ (2): use the FTC.

(2) $\implies$ (1): This is the nontrivial bit. Fix $z_0 \in U$ and define

$$F(z) := \int_{z_0}^{z} f(w)\,dw := \int_\gamma f(w)\,dw$$

for any γ connecting z_0 to z. By (2), the choice doesn't matter, and so F is well-defined; while as U is connected, F is defined on all of U.

To show that F is *holomorphic* with $F' = f$, we must prove that

$$\lim_{h \to 0} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| = 0 \qquad \text{(II.A.12)}$$

for any fixed z. By (2),

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f\,dz - \int_{z_0}^{z} f\,dz = \int_{z}^{z+h} f\,dz,$$

where the integral from z to $z+h$ is taken along a line segment ℓ_h. (Since U is open, we can take h sufficiently small that this is possible.) We may also write $f(w) = f(z) + \varepsilon(w)$ with $\lim_{w\to z}\varepsilon(w) = 0$, because $f \in \mathcal{C}^0(U)$. Putting these together, we have

$$\begin{aligned}
\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| &= \left|\frac{1}{h}\int_z^{z+h} f(w)\,dw \;-\; f(z)\right| \\
&= \left|\frac{1}{h}\int_z^{z+h} f(z)\,dw \;-\; f(z) \;+\; \frac{1}{h}\int_z^{z+h}\varepsilon(w)\,dw\right| \\
&= \left|\frac{1}{h}\int_z^{z+h}\varepsilon(w)\,dw\right| \\
&\le \left|\frac{1}{h}\right| L(\ell_h)\|\varepsilon(w)\|_{\ell_h} = \|\varepsilon(w)\|_{\ell_h} \to 0
\end{aligned}$$

as $h \to 0$, which yields (II.A.12). $\square$

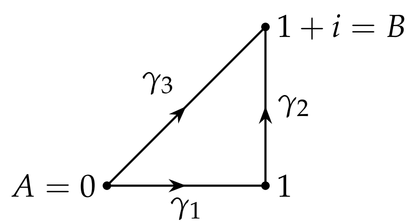
II.A.13. Basic Examples. Here are some functions with holomorphic primitive:

- on $\mathbb{C}$: z^n ($n \ge 0$), e^z, $\sin z$, $\cos z$
- on $\mathbb{C}^*$: z^n ($n \le -2$)
- on D_r: any function given by a power series at 0 w/ radius of convergence r

All have trivial loop-integrals about the origin.

II.A.14. Basic Non-Example. Consider $f(z) = \bar{z}$, and the three paths $\gamma_i\colon [0,1] \to \mathbb{C}$ defined by

$$\begin{cases} \gamma_1(t) = t \\ \gamma_2(t) = 1 + \mathbf{i}t \\ \gamma_3(t) = (1+\mathbf{i})t \end{cases}$$



Now $\int_{\gamma_i} \overline{z}\, dz = \int_0^1 \overline{\gamma_i(t)} \gamma_i'(t) dt$, so

$$\int_{\gamma_1} \overline{z}\, dz = \int_0^1 t \cdot 1 dt = \left.\frac{t^2}{2}\right|_0^1 = \frac{1}{2}$$

$$\int_{\gamma_2} \overline{z}\, dz = \int_0^1 (1 - \mathbf{i}t) \cdot \mathbf{i} dt = \mathbf{i} \int_0^1 dt + \int_0^1 t\, dt = \frac{1}{2} + \mathbf{i}$$

$$\int_{\gamma_3} \overline{z}\, dz = \int_0^1 (1 - \mathbf{i})t \cdot (1 + \mathbf{i}) dt = 2 \int_0^1 t\, dt = t^2|_0^1 = 1$$

$$\implies \quad \int_{\gamma_1 + \gamma_2} \overline{z}\, dz = \int_{\gamma_1} \overline{z}\, dz + \int_{\gamma_2} \overline{z}\, dz = \frac{1}{2} + \left(\frac{1}{2} + \mathbf{i}\right) = 1 + \mathbf{i}$$
$$\neq 1 = \int_{\gamma_3} \overline{z}\, dz.$$

We conclude with a preliminary result on loop integrals on a punctured disk D_r^*. To prove it, we'll need the following:

II.A.15. LEMMA. *Let $S \subseteq \mathbb{C}$ be any subset, γ a path in S, and $\{f_k\}$ a sequence of functions in $C^0(S)$. If $\sum_{k=0}^{\infty} f_k$ converges uniformly on S, or even just on γ, then*

$$\sum_{k=0}^{\infty} \left(\int_\gamma f_k(z) dz \right) = \int_\gamma \left(\sum_{k=0}^{\infty} f_k \right) dz.$$

PROOF. We have

$$\left| \sum_{k=0}^{n} \int_\gamma f_k dz - \int_\gamma f dz \right| = \left| \int_\gamma \left(\sum_{k=0}^{n} f_k \right) dz - \int_\gamma f dz \right|$$
$$= \left| \int_\gamma \left(\sum_{k=0}^{n} f_k - f \right) dz \right|$$
$$\leq L(\gamma) \left\| \sum_{k=0}^{n} f_k - f \right\|_\gamma \xrightarrow{n \to \infty} 0$$

since the series converges uniformly on γ. $\square$

II.A.16. PROPOSITION. *Let f be given on $S := \overline{D}_r \setminus \{0\}$ by the Laurent series*

$$(f(z) :=)\ \frac{a_{-m}}{z^m} + \cdots + \frac{a_{-1}}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

where the radius of convergence of the power series part is $> r$. Then

$$\int_{\partial D_r} f(z)dz = 2\pi\mathbf{i}a_{-1}.$$

PROOF. The power series part converges uniformly on $\overline{D}_r$ to a continuous function, so (writing $\gamma = \partial D_r$)

$$\int_\gamma \sum_{k=0}^{\infty} a_k z^k dz = \sum_{k=0}^{\infty} a_k \int_\gamma z^k dz = 0$$

since γ is closed and z^k has primitive $\frac{z^{k+1}}{k+1}$ on $D_{r+\epsilon}$. For $m \neq -1$, the other integrals are also 0; but (parametrizing ∂D_r by $z = \gamma(t) := re^{2\pi\mathbf{i}t}$, $t \in [0,1]$)

$$\int_\gamma \frac{1}{z}dz = \int_0^1 \frac{1}{re^{2\pi\mathbf{i}t}} 2\pi\mathbf{i}re^{2\pi\mathbf{i}t}dt = 2\pi\mathbf{i}\int_0^1 dt = 2\pi\mathbf{i}. \qquad \square$$

Exercises

(1) Find $\int_{|z|=r} x\,dz$, where the circle is traversed counterclockwise. (Here $x = \mathrm{Re}(z)$.)

(2) Let $U \subseteq \mathbb{C}$ be open, $f \in \mathscr{Hol}(U)$, and γ a closed $\mathcal{C}^1$ path in U. Show that $\int_\gamma \overline{f(z)}f'(z)\,dz$ is purely imaginary. (You may assume f is continuous.)

(3) Compute $\int_{|z|=1} |z-1|\,|dz|$.

(4) If $P(z)$ is a polynomial and C denotes the circle $|z-a| = R$, find the value of $\int_C P(z)d\bar{z}$.

(5) Let σ be a vertical segment, parametrized by $\sigma(t) = z_0 + \mathbf{i}tc$, $t \in [-1,1]$, where z_0 is a fixed complex number, and c is a fixed positive real number. Let $\alpha = z_0 + x$ and $\alpha' = z_0 - x$, where $x \in \mathbb{R}_{>0}$. Find $\lim_{x\to 0} \int_\sigma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right) dz$.

(6) Integrate $\sqrt{z}$ along the upper unit semicircle, oriented counterclockwise, in two different ways: (a) by the definition; and (b) using a primitive for $\sqrt{z}$ on $\mathbb{C} \setminus \{\mathbf{i}r \mid r \leq 0\}$.