

II.B. Prelude to Cauchy

§ Cauchy's actual theorem.

In the coming chapters, we are aiming at proving from scratch various versions of *Cauchy's Theorem*. Informally, it says that if $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$, with U a simply connected region, then $\int_{\gamma} f dz = 0$ for γ closed. We want simple connectivity (U has no “holes”) here so that the region enclosed by γ is automatically in U .

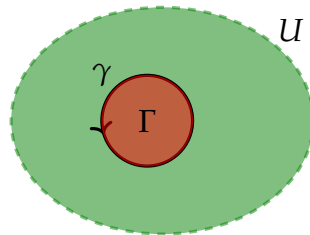
If we assume that f is not just $\mathbb{C}$ -differentiable but *continuously* so,² this can be done using **Green's Theorem** from vector calculus (hence not quite from scratch): let $\vec{F}(x, y)$ be a vector field of class $\mathcal{C}^1$ on U . Then

$$(II.B.1) \quad \int_{\gamma} \vec{F} \cdot d\vec{\ell} = \iint_{\Gamma} \text{curl}(\vec{F}) dx dy$$

for γ oriented counterclockwise and

$\partial\Gamma = \gamma$. To make this explicit, writing

$\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and $\vec{\gamma}(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \end{pmatrix}$, we have $\text{curl}(\vec{F}) := \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$ and $d\vec{\ell} := \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \end{pmatrix} dt$. Going from $\mathbb{R}^2$ to $\mathbb{C}$, notation shifts to $dz = \gamma'(t)dt = \gamma_1'(t)dt + i\gamma_2'(t)dt = dx + i dy$ and



$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} (u + iv)(dx + i dy) \\ &= \int_{\gamma} (u dx - v dy) + i \int_{\gamma} (v dx + u dy) \\ &= \int_{\gamma} \begin{pmatrix} u \\ -v \end{pmatrix} \cdot d\vec{\ell} + i \int_{\gamma} \begin{pmatrix} v \\ u \end{pmatrix} \cdot d\vec{\ell} \\ &\stackrel{(II.B.1)}{=} \iint_{\Gamma} - \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) dx dy + i \iint_{\Gamma} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy = 0, \end{aligned}$$

where the last step is by the Cauchy-Riemann equations. This is essentially Cauchy's original result, but due to more recent improvements in technology, it will not be our “official” Cauchy Theorem.

²by which we mean that u_x, u_y, v_x, v_y are $\mathcal{C}^0$. This is (as an assumption) undesirable: while it will turn out to be the case that $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U) \implies f \in \mathcal{A}\mathcal{N}(U)$ (a fortiori continuously differentiable), it will be *Cauchy* that enables us to show this!!

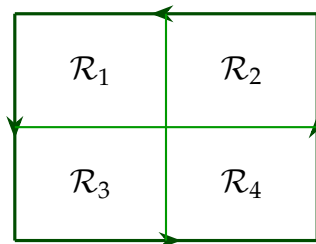
§ Goursat's theorem.

Here is one of those technical improvements: we only assume that the complex derivative *exists* at each point. (This is significant, because we eventually want to prove that continuous differentiability *follows* from differentiability.)

II.B.2. THEOREM (Goursat, 1884). *Given a closed rectangle $\mathcal{R} \subset \mathbb{C}$ and $f \in \mathcal{H}\mathcal{O}\mathcal{L}(\mathcal{R})$,³*

$$\int_{\partial\mathcal{R}} f dz = 0.$$

PROOF. Referring to the figure



we have $\mathcal{R} = \bigcup_{i=1}^4 \mathcal{R}_i \implies \int_{\partial\mathcal{R}} f dz = \sum_{i=1}^4 \int_{\partial\mathcal{R}_i} f dz \implies$

$$\begin{aligned} \left| \int_{\partial\mathcal{R}} f dz \right| &\leq \sum_{i=1}^4 \left| \int_{\partial\mathcal{R}_i} f dz \right| \quad [\text{Let } \mathcal{R}^{(1)} := \text{the } \mathcal{R}_i \text{ with largest } |\int_{\partial\mathcal{R}_i}|] \\ &\leq 4 \left| \int_{\partial\mathcal{R}^{(1)}} f dz \right| \\ &\leq 4^2 \left| \int_{\partial\mathcal{R}^{(2)}} f dz \right| \quad [\text{subdivide } \mathcal{R}^{(1)} \text{ and repeat}] \\ &\leq \dots \\ &\leq 4^n \left| \int_{\partial\mathcal{R}^{(n)}} f dz \right|. \end{aligned}$$

The sequence $z_i := \text{center}(\mathcal{R}^{(i)})$ is Cauchy (why?), hence has a limit point which lies in every $\mathcal{R}^{(n)}$ (since they are compact and contain the tails). The intersection cannot contain more than one point and

³Recall that this means that there exists an open $U \supset \mathcal{R}$ such that f extends to a holomorphic function on U .

so

$$\bigcap_{i=1}^{\infty} \mathcal{R}^{(i)} = \lim_{i \rightarrow \infty} z_i =: z_{\infty}.$$

Now $f(z) = f(z_{\infty}) + f'(z_{\infty})(z - z_{\infty}) + (z - z_{\infty})H(z)$, where

$$H(z) = \begin{cases} \frac{f(z) - f(z_{\infty})}{z - z_{\infty}} - f'(z_{\infty}), & z \neq z_{\infty} \\ 0, & z = z_{\infty} \end{cases}$$

is continuous by the definition of complex differentiability (of f). Since 1 and z have holomorphic primitives on all of $\mathbb{C}$, Proposition II.A.10 gives

$$\begin{aligned} \int_{\partial \mathcal{R}^{(n)}} f(z) dz &= \int_{\partial \mathcal{R}^{(n)}} f(z_{\infty}) dz + f'(z_{\infty}) \int_{\partial \mathcal{R}^{(n)}} (z - z_{\infty}) dz \\ &\quad + \int_{\partial \mathcal{R}^{(n)}} (z - z_{\infty}) H(z) dz. \end{aligned}$$

So

$$\begin{aligned} \left| \int_{\partial \mathcal{R}} f dz \right| &\leq 4^n \left| \int_{\partial \mathcal{R}^{(n)}} (z - z_{\infty}) H(z) dz \right| \\ &\leq 4^n L(\partial \mathcal{R}^{(n)}) \|z - z_{\infty}\|_{\partial \mathcal{R}^{(n)}} \|H(z)\|_{\partial \mathcal{R}^{(n)}} \\ &= 4^n \cdot \frac{1}{2^n} L(\partial \mathcal{R}) \cdot \frac{1}{2^n} \text{diam}(\mathcal{R}) \cdot \|H(z)\|_{\partial \mathcal{R}^{(n)}} \\ &= \text{const.} \times \|H(z)\|_{\partial \mathcal{R}^{(n)}} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

since the $\mathcal{R}^{(n)}$ are contained in smaller and smaller neighborhoods of z_{∞} in U . $\square$

The result from the last Chapter used above (Prop. II.A.10) stated that for a *continuous* function f on a region U , the conditions

$$(II.B.3) \quad \begin{cases} f \text{ has a holomorphic primitive } F \\ \int_A^B f dz \text{ is independent of path} \\ \int_{\gamma} f dz = 0 \text{ for every loop } \gamma \subset U \end{cases}$$

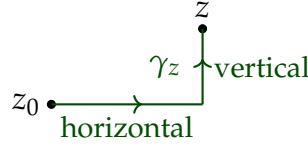
were equivalent. We will now put this together with Goursat's theorem to show that every holomorphic function on the disk has a holomorphic primitive.

II.B.4. PROPOSITION. Take U to be the disk $D(z_0, r)$.

- (a) If $f \in C^0(U)$ has $\int_{\partial\mathcal{R}} f dz = 0$ for all rectangles $\mathcal{R} \subset U$, then the conditions (II.B.3) hold.
- (b) If $f \in \mathcal{Hol}(U)$, then the conditions (II.B.3) hold.

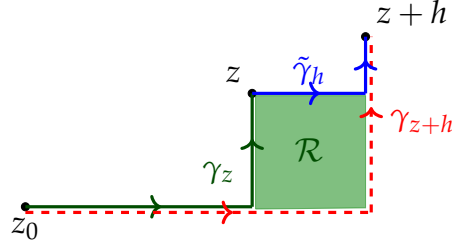
PROOF. Once we know (a), (b) is an immediate corollary of Goursat. To prove (a), we will construct a holomorphic primitive F for f .

Given $z \in U$, let γ_z be the path



and set $F(z) := \int_{\gamma_z} f(w) dw$. Then

$$\gamma_{z+h} - \gamma_z = \partial\mathcal{R} + \tilde{\gamma}_h$$



together with $\int_{\partial\mathcal{R}} f(w) dw = 0$ gives

$$F(z+h) - F(z) = \int_{\tilde{\gamma}_h} f(w) dw.$$

Trivially $f(z) = \frac{1}{h} \int_{\tilde{\gamma}_h} f(z) dw$ (z constant), and so

$$\begin{aligned} \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \left| \frac{1}{h} \int_{\tilde{\gamma}_h} (f(w) - f(z)) dw \right| \\ &\leq \frac{L(\tilde{\gamma}_h)}{|h|} \|f(w) - f(z)\|_{\tilde{\gamma}_h} \\ &\leq \sqrt{2} \|f(w) - f(z)\|_{\tilde{\gamma}_h} \rightarrow 0 \quad \text{with } h. \quad \square \end{aligned}$$

§ Remark on differential forms.

Recall that the change-of-variable formula for multiple integrals implies that given $F(x, y) = u + iv \in \mathcal{C}^1(U)$ and $K \subset U$ compact,

$$(II.B.5) \quad \iint_{F(K)} du \, dv = \iint_K \underbrace{(u_x v_y - u_y v_x)}_{\det(J_F)} dx \, dy.$$

If $F \in \mathcal{H}ol(U)$ then $u_x v_y - u_y v_x = u_x^2 + v_x^2 = |F'(z)|^2$ by Cauchy-Riemann,⁴ and then the above equals $\iint_K |F'(z)|^2 dx \, dy$.

The best way to look at this change-of-variable computation is in terms of **differential forms**. Let $I = \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ stand for multi-indices with $i_1 < \dots < i_k$. Then for $U \subset \mathbb{R}^n$, we define⁵

$$\begin{aligned} A^k(U) &:= \left\{ \text{expressions of the form} \left| \sum_I f_I(\underline{x}) dx_{i_1} \wedge \dots \wedge dx_{i_k} \right| f_I \in C^\infty(U) \right\} \\ &\cong C^\infty(U) \otimes \wedge^k \mathbb{R}^n. \end{aligned}$$

Here you should think of “ $\wedge$ ” as an anti-commutative version of tensor product: so $A \wedge B = -B \wedge A$, and $\wedge^k \mathbb{R}^n$ is a real vector space of dimension $\binom{n}{k}$ with basis consisting of the $e_{i_1} \wedge \dots \wedge e_{i_k}$.

The differential k -forms $A^k(U)$ generalize functions on U in a useful way. There is an **exterior derivative**

$$A^k(U) \xrightarrow{d} A^{k+1}(U)$$

which is given by $\mathbb{C}$ -linearly extending the rule

$$d\{f(\underline{x}) dx_{i_1} \wedge \dots \wedge dx_{i_k}\} := \sum_{l \in \{1, \dots, n\} \setminus I} \frac{\partial f}{\partial x_l} dx_l \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

The k -dimensional generalization of a $\mathcal{C}^1$ path is a “ k -chain” — this is what you want to integrate a k -form over. These come with a **boundary map**

$$C_{k+1}(U) \xrightarrow{\partial} C_k(U).$$

from $(k+1)$ -chains to k -chains.

⁴You also need to use that F holomorphic $\implies \frac{dF}{dz} = \frac{\partial F}{\partial x} = u_x + iv_x$.

⁵The C^∞ functions here are complex valued.

Say $\omega \in A^k(U)$ and $\gamma \in C_{k+1}(U)$. Goursat was the first to write down the multi-variable Fundamental Theorem of Calculus (a.k.a. **Stokes's Theorem**) in the form

$$(II.B.6) \quad \int_{\gamma} d\omega = \int_{\partial\gamma} \omega.$$

This contains the classical Green, Gauss, and Stokes theorems and is obviously a powerful generalization. If ω is an expression of the form $f(z)dz$, then it relates a contour integral of an arbitrary smooth (or even C^1) function to an area integral of its “curl”.

Going back to (II.B.5), the point is that working in $C_{\mathbb{R}}^{\infty}(U) \otimes \wedge^2 \mathbb{R}^2$ we have

$$\begin{aligned} du \wedge dv &= (u_x dx + u_y dy) \wedge (v_x dx + v_y dy) \\ &= u_x v_x dx \wedge dx + u_x v_y dx \wedge dy + u_y v_x dy \wedge dx + u_y v_y dy \wedge dy \\ &= (u_x v_y - u_y v_x) dx \wedge dy, \end{aligned}$$

since $dy \wedge dy = -dy \wedge dy$ forces it to be zero, and so forth. The Jacobian determinant thus arises naturally out of the algebra of differential forms; the general reason for this is the tight relationship between determinants and top exterior powers of a vector space.

The best place to get acquainted with differential forms is Rudin's “Principles of Mathematical Analysis” (a.k.a. “baby Rudin”).