

II.C. Cauchy's Theorem (part 1)

§ Local Cauchy theorem.

In the last chapter, we proved that for a disk D

$$(II.C.1) \quad \left. \begin{array}{l} f \in \mathcal{H}ol(D) \\ \gamma \subset D \text{ loop} \end{array} \right\} \implies \int_{\gamma} f dz = 0$$

What follows is a refinement of this result.

II.C.2. LEMMA. *Let $\mathcal{R} \subset \mathbb{C}$ be a closed rectangle, $z_1, \dots, z_n \in \mathcal{R}$, and $f \in \mathcal{H}ol(\mathcal{R} \setminus \{z_1, \dots, z_n\})$ satisfy*

$$\lim_{z \rightarrow z_j} (z - z_j)f(z) = 0 \quad (\forall j).$$

Then $\int_{\partial \mathcal{R}} f(z) dz = 0$.

PROOF. Let $\epsilon > 0$ be given, and take δ sufficiently small that

$$|z - z_j| < \sqrt{2}\delta \implies |(z - z_j)f(z)| < \frac{\epsilon}{n}.$$

Let $\mathcal{R}_j$ be a square centered at z_j of side length 2δ , and decompose $\mathcal{R}$ into sub-rectangles (not necessarily of the same size) including these.

By Goursat's theorem, the integrals over the boundaries of rectangles not containing a z_j are zero, and thus

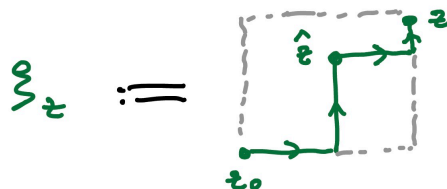
$$\begin{aligned} \left| \int_{\partial \mathcal{R}} f dz \right| &= \left| \sum_{j=1}^n \int_{\partial \mathcal{R}_j} f dz \right| \\ &\leq \sum_{j=1}^n L(\partial \mathcal{R}_j) \|f\|_{\partial \mathcal{R}_j} \\ &= \sum_{j=1}^n 8\delta \left\| \frac{(z - z_j)f(z)}{(z - z_j)} \right\|_{\partial \mathcal{R}_j} \\ &< \frac{8\delta\epsilon}{n} \sum_{j=1}^n \left\| \frac{1}{z - z_j} \right\|_{\partial \mathcal{R}_j} = \frac{8\delta\epsilon}{n} \frac{n}{\delta} \\ &= 8\epsilon. \end{aligned}$$

Since ϵ was arbitrary, the integral is zero. □

Writing $D = D(z_0, r)$ and $U = D \setminus \{z_1, \dots, z_n\}$, we then have the

II.C.3. THEOREM. Suppose $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$ has $\lim_{z \rightarrow z_j} (z - z_j)f(z) = 0$ for each j . Then for every closed C^1 path $\gamma \subset U$, we have $\int_\gamma f(z)dz = 0$.

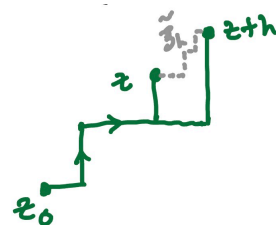
PROOF. As in the proof of Prop. II.B.4, construct a primitive for f on U : given $z \in U$, let



for *any* choice of $\hat{z}$ within the (closed) dashed box, subject only to the condition that the path avoid all the $\{z_j\}_{j=1}^n$. The difference between any two choices consists of a sum of boundaries of boxes (chosen to avoid $\{z_j\}_{j=1}^n$). By the Lemma,

$$F(z) := \int_{\zeta_z} f(z)dz$$

is well-defined. Once again, one shows it is holomorphic (with derivative f) by computing



$$F(z+h) - F(z) = \sum_i \int_{\partial R_i} f(w)dw + \int_{\tilde{\zeta}_h} f(w)dw$$

and the proof proceeds as before. $\square$

§ Homotopy Cauchy theorem.

Let $U \subset \mathbb{C}$ be an open set, and $\gamma \subset U$ a $\mathcal{C}^0$ path: i.e., $\gamma: [a, b] \rightarrow U$ is of class $\mathcal{C}^0$. For each $z \in U$, $\exists \epsilon > 0$ such that $D(z, \epsilon) \subset U$. Hence $d(z, U^c) := \inf_{w \in U^c} |z - w| > 0$ defines a (continuous!) function

$$d: U \rightarrow \mathbb{R}_+.$$

Since $[a, b]$ is compact, $d \circ \gamma$ (also continuous!) *achieves* its greatest lower bound; call it r . This r is a *value taken by* $d \circ \gamma$, hence lies in $\mathbb{R}_+$, with the consequence that

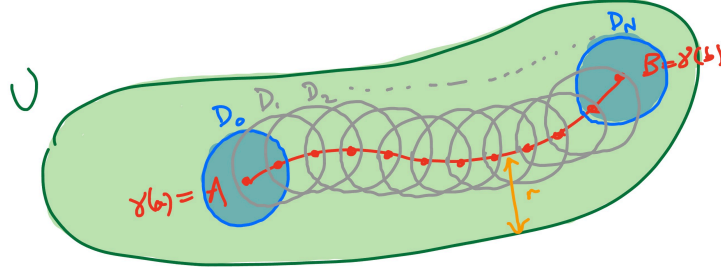
$$(II.C.4) \quad D(\gamma(t), r) \subset U \text{ for every } t \in [a, b].$$

Compactness of $[a, b]$ also tells us that γ is uniformly continuous, i.e. given $\epsilon \in (0, \frac{r}{2})$, $\exists \delta > 0$ such that for any $s, t \in [a, b]$

$$|s - t| < \delta \implies |\gamma(s) - \gamma(t)| < \epsilon.$$

Let $\Pi = \{a = t_0 < t_1 < \dots < t_N = b\}$ be a partition of $[a, b]$ with $|t_i - t_{i-1}| < \delta$ ($\forall i$). Writing $I_i = [t_i, t_{i+1}]$, we have

$$\gamma(I_i) \subset D(\gamma(t_i), \epsilon) =: D_i \subset U.$$



Now keep in mind that γ is $\mathcal{C}^0$, and is *not even assumed rectifiable*.

II.C.5. DEFINITION. Given $f \in \text{Hol}(U)$, let

- Π be a partition of $[a, b]$ into intervals I_i ,
- $D_i \subset U$ be open disks with $\gamma(I_i) \subset D_i$, and
- $F_i \in \mathcal{H}ol(D_i)$ be primitives of $f|_{D_i}$ (which exist by (II.C.1)).

and set

$$(II.C.6) \quad \int_{\gamma} f(z) dz := \sum_{i=0}^{N-1} (F_i(\gamma(t_{i+1})) - F_i(\gamma(t_i))).$$

Notice that this is a new *definition* of the integral, so two questions immediately arise:

Q1 If γ is rectifiable, does (II.C.6) give the right answer?

Yes: this is essentially the fundamental theorem of calculus. Referring to the beginning of Chapter II.A for the definition of the integral for rectifiable paths, it suffices to check the case where U is a disk and F a global primitive:

$$\begin{aligned} & \lim_{|\Pi| \rightarrow 0} \left| \sum_i f(\gamma(t_i))(\Delta\gamma)_i - (F(B) - F(A)) \right| \\ & \leq \lim_{|\Pi| \rightarrow 0} \sum_i |f(\gamma(t_i))(\Delta\gamma)_i - (F(t_{i+1}) - F(t_i))| \\ & = \lim_{|\Pi| \rightarrow 0} \sum_i |(\Delta\gamma)_i| |\varepsilon_i| = 0, \end{aligned}$$

where $(\Delta\gamma)_i := \gamma(t_{i+1}) - \gamma(t_i)$, $F(t_{i+1}) - F(t_i) = (\Delta\gamma)_i(f(t_i) + \varepsilon_i)$, and $\varepsilon_i \rightarrow 0$ with $|\Pi|$ (uniformly in i).

Q2 If γ isn't rectifiable,⁶ is (II.C.6) well-defined?

Yes again! Here is a sketch of the argument:

- Independence of choice of the local primitives $\{F_i\}$ is trivial.
- Independence of choice of the $\{D_i\}$ is too (if the $\{I_i\}$ are fixed).
- The issue is what happens when there are two different partitions Π, Π' . Since any two have a common refinement, we must check that (II.C.6) is invariant under refinement. We can do this on one interval I_i and choose any D, F we like:



here we just take D, F to be D_i, F_i for both $[t_i, c]$ and $[c, t_{i+1}]$, so (II.C.6) doesn't change.

⁶For instance, $\gamma(t) = t + it \sin(\frac{1}{t})$, $t \in [0, 1]$.

II.C.7. DEFINITION. Paths $\gamma, \eta: [a, b] \rightarrow U$ are **close together** if there exist a partition Π of $[a, b]$ and disks $D_i \subset U$ such that

$$\gamma(I_i), \eta(I_i) \subset D_i \quad (\forall i).$$

In the free abelian group on points of $\mathbb{C}$, write

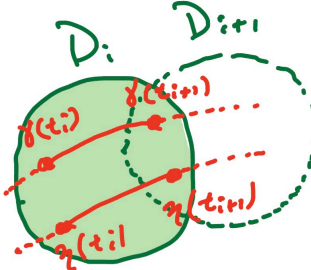
$$\partial\gamma := [\gamma(b)] - [\gamma(a)].$$

This is $0 \iff \gamma$ is closed.

II.C.8. LEMMA. Let $\gamma, \eta \subset U$ be close together, with $\partial\gamma = \partial\eta$, and $f \in \mathcal{H}ol(U)$. Then

$$\int_{\gamma} f dz = \int_{\eta} f dz$$

PROOF. CASE 1: $\partial\gamma, \partial\eta \neq 0$. This forces γ and η to have the same starting point and same ending point. Referring to the figure,



$$\begin{aligned} \int_{\gamma} f dz - \int_{\eta} f dz &= \sum_{i=0}^{n-1} \{ [F_i(\gamma(t_{i+1})) - F_i(\gamma(t_i))] - [F_i(\eta(t_{i+1})) - F_i(\eta(t_i))] \} \\ &= \sum_{i=0}^{n-1} \{ [F_i(\gamma(t_{i+1})) - F_i(\eta(t_{i+1}))] - [F_i(\gamma(t_i)) - F_i(\eta(t_i))] \}. \end{aligned}$$

At this point it is harmless to change the first two F_i 's to F_{i+1} , as F_i and F_{i+1} only differ by a constant on the overlap. This puts the sum in a form that telescopes:

$$\begin{aligned} &= \sum_{i=0}^{n-1} \{ [F_{i+1}(\gamma(t_{i+1})) - F_{i+1}(\eta(t_{i+1}))] - [F_i(\gamma(t_i)) - F_i(\eta(t_i))] \} \\ &= F_n(\gamma(b)) - F_n(\eta(b)) - F_0(\gamma(a)) + F_0(\eta(a)) \\ &= 0 \quad (\text{since } \gamma(b) = \eta(b) \text{ and } \gamma(a) = \eta(a)). \end{aligned}$$

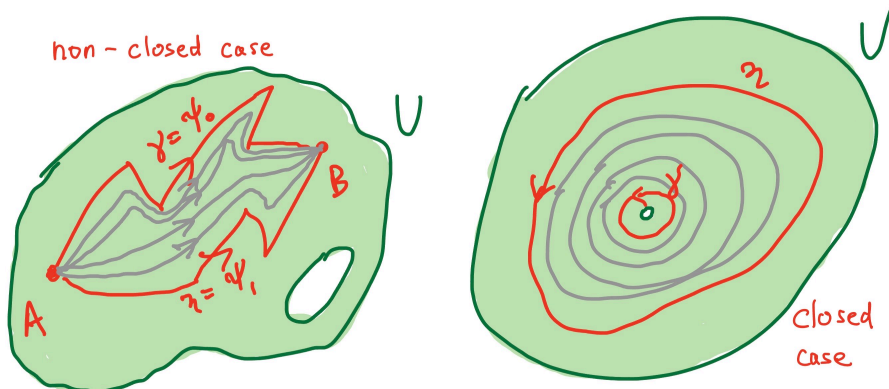
CASE 2: $\partial\gamma = 0 = \partial\eta$. Then $\gamma(a)$ [resp. $\gamma(b)$] may *not* equal $\eta(a)$ [resp. $\eta(b)$], but $\gamma(a) = \gamma(b)$ and $\eta(a) = \eta(b)$, and we can take $F_n = F_0, D_n = D_0$. So the same calculation as in Case 1 works, except the terms cancel in different pairs at the end. $\square$

As above, let $\gamma, \eta: [a, b] \rightarrow U$ be $\mathcal{C}^0$ paths with $\partial\gamma = \partial\eta$.

II.C.9. DEFINITION. γ is **homotopic** to η in U , written $\gamma \sim \eta$, if there is a $\mathcal{C}^0$ map

$$\psi: [a, b] \times [0, 1] \rightarrow U$$

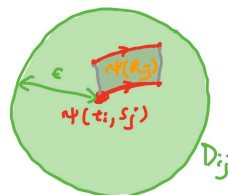
with $\psi(t, 0) = \gamma(t)$, $\psi(t, 1) = \eta(t)$, and (writing $\psi(t, s) =: \psi_s(t)$) $\partial\psi_s = \partial\gamma (= \partial\eta) \forall s \in [0, 1]$.



II.C.10. THEOREM. Given γ, η homotopic and $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$, we have

$$\int_{\gamma} f dz = \int_{\eta} f dz.$$

PROOF. Let Π, P be partitions of $[a, b]$ resp. $[0, 1]$ and write I_i, J_j for the subintervals and $\mathcal{R}_{ij} := I_i \times J_j$. By uniform continuity there is an $\epsilon \in (0, d(\psi(\mathcal{R}), U^c))$ such that $\psi(\mathcal{R}_{ij}) \subset D(\psi(t_i, s_j), \epsilon) =: D_{ij} (\forall i, j)$. Clearly, ψ_{s_j} and $\psi_{s_{j+1}}$ are close together for each j , and have the same “boundary” $\partial(\cdot)$. So by the Lemma, we have that $\int_{\psi_0=s_0} f dz = \int_{\psi_1=s_1} f dz = \dots = \int_{\psi_1=s_n} f dz$. This does the job since $\psi_0 = \gamma$ and $\psi_1 = \eta$. $\square$



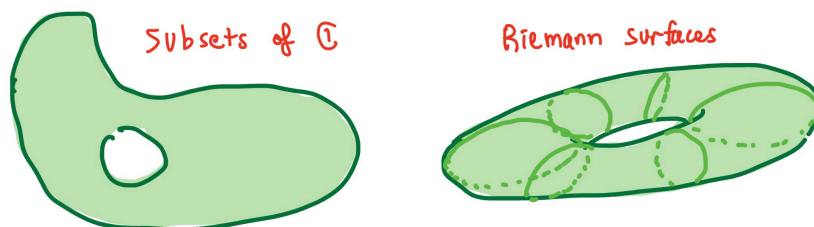
§ Simple connectivity.

Let $U \subset \mathbb{C}$ be open.

II.C.11. DEFINITION. U is **simply connected** if every $\mathcal{C}^0$ closed path $\gamma \subset U$ is homotopic (in U) to a *point*, i.e. constant path.

Of course, this definition extends to Riemann surfaces as well as manifolds of any dimension.

II.C.12. NON-EXAMPLES. There are definitely loops that you cannot contract to a point in the following pictures:



More specifically, $\mathbb{C}^*$, D_r^* , $\mathbb{C} \setminus [-1, 1]$, and the Riemann surface of $\sqrt{(z-a)(z-b)(z-c)}$ are not simply connected.

II.C.13. EXAMPLES. The following are simply connected:

- (1) $\mathbb{C}$ itself, or $\hat{\mathbb{C}}$ if you want something compact.
- (2) Convex subsets: if $z, w \in U$, then $sz + (1-s)w \in U$ ($\forall s \in [0, 1]$).
In other words, U is convex if it always contains the segment connecting any two of its points.
- (3) Star-shaped sets: These are more general than convex subsets. U is **star-shaped** if there is a $z_0 \in U$ such that U contains the segments connecting z_0 to every other point of U .
- (4) A disk is convex and star-shaped; $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is star-shaped (with $z_0 = 1$), as is $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$ (with $z_0 = 0$).

PROOF THAT $\mathbb{C}$, CONVEX SETS, AND $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ SATISFY II.C.11.

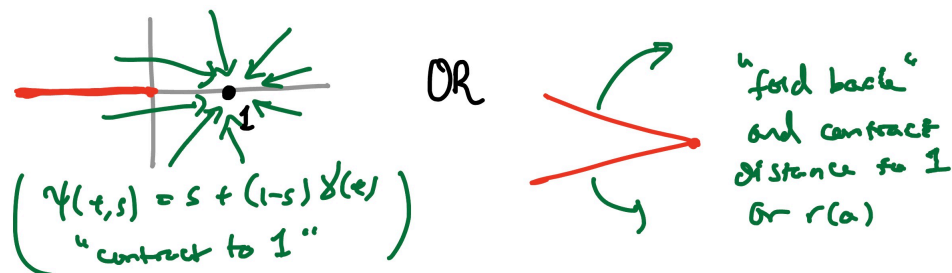
(i) Given $\gamma: [a, b] \rightarrow \mathbb{C}$ closed, define $\psi: [a, b] \times [0, 1] \rightarrow \mathbb{C}$ by

$$\psi(t, s) := (1-s) \cdot \gamma(t) \quad (\text{"contraction to 0"}).$$

(ii) Let U be convex. We can contract γ to its own starting (or ending) point:

$$\psi(t, s) := s \cdot \gamma(a) + (1 - s) \cdot \gamma(t).$$

(iii) For $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ there are two methods:

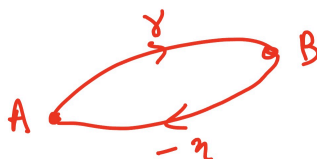


For the right-hand picture, writing $\gamma(t) = r(t)e^{i\theta(t)}$, we put

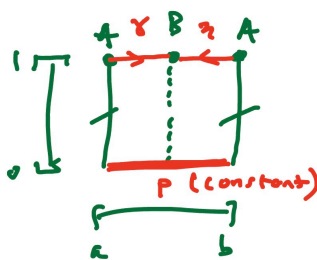
$$\psi(t, s) := (s \cdot r(a) + (1 - s) \cdot r(t))e^{i\theta(t)(1-s)}.$$

The key point is: $\theta(a) = \theta(b)$, not just mod $2\pi\mathbb{Z}$, because $\gamma(t)$ can't go around 0. $\square$

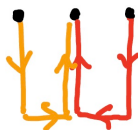
II.C.14. REMARK. Homotopy of paths based at A and B is clearly transitive. Furthermore, if U is simply connected, then for $\gamma \sim \eta$ (with $\partial\gamma = [B] - [A] = \partial\eta$),



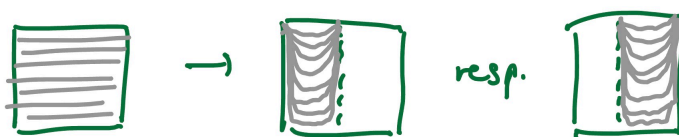
is closed and homotopic to a point p . The homotopy ψ can be visualized as follows:



Notice that ψ takes the same value on the marked vertical segments, since $\partial\psi_s = 0$ for every s . Thus ψ of these two paths in the figure above



have precisely the same image, call it $\mathcal{P}$. Composing ψ with a function from $[0, 1]^2$ to $[a, b] \times [0, 1]$ sending



gives $\gamma \sim \mathcal{P} \sim \eta$.

We conclude that in a simply-connected U , $\partial\gamma = \partial\eta \implies \gamma \sim \eta$.

Why are we doing this? To get the following corollaries to the “homotopy Cauchy Theorem” II.C.10.

II.C.15. COROLLARY. If U is simply connected, $\gamma \subset U$ is closed, and $f \in \mathcal{H}ol(U)$, then $\int_{\gamma} f dz = 0$.

PROOF. In U any closed path γ is homotopic to a point (i.e. constant path η), and $\int_{\eta} f dz = 0$ since $\eta'(t) = 0$. $\square$

II.C.16. COROLLARY. U simply connected $\implies$ any $f \in \mathcal{H}ol(U)$ has a global holomorphic primitive on U , namely

$$F(z) := \int_{z_0}^z f(w) dw \quad (+ \text{const.}).$$

PROOF. F is well-defined because any two paths in U from z_0 to z are homotopic in U by the remark. (Hence we may apply homotopy Cauchy directly.) That F is differentiable with $F' = f$ is as usual. $\square$

Exercises

(1) Prove that every star-shaped set is simply connected.