

II.D. Some interesting functions

§ The complex logarithm.

We begin by giving an approach to $\log(z)$ which is complementary to that in Chapter I.F.

Recall that $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ is star-shaped and therefore simply-connected, and that *on a simply-connected region, every holomorphic function has a global holomorphic primitive* (Cor. II.C.16). Therefore, on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, the function $\frac{1}{z}$ has a global primitive

$$\log(z) := \int_1^z \frac{1}{w} dw,$$

where the integral is carried out on any path from 1 to z in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Properties:

- $\log(z) \in \mathcal{Hol}(\mathbb{C} \setminus \mathbb{R}_{\leq 0})$
- $\text{Im}(\log(z)) \in (-\pi, \pi)$
- $\log(z)$ is analytic: given $z_1 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, write

$$\log(z) = \underbrace{\int_1^{z_1} \frac{dw}{w}}_{\text{constant}} + \underbrace{\int_{z_1}^z \frac{dw}{w}}_{\text{power series}}$$

To explain how the second term is realized as a power series, let w_1 be such that $e^{w_1} = z_1$ and $\text{Im}(w_1) \in (-\pi, \pi)$. Put

$$F_{[z_1]}(z) := w_1 + \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \frac{(z - z_1)^n}{z_1^n}$$

on $D(z_1, |z_1|)$.



Differentiation (and setting $m = n - 1$) yields

$$\begin{aligned} F'_{[z_1]}(z) &= \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \cdot n \cdot \frac{(z - z_1)^{n-1}}{z_1^n} = \frac{1}{z_1} \sum_{m \geq 0} (-1)^m \left(\frac{z - z_1}{z_1} \right)^m \\ &= \frac{1}{z_1} \cdot \frac{1}{1 + \frac{z - z_1}{z_1}} = \frac{1}{z_1 + z - z_1} = \frac{1}{z} = \log'(z), \end{aligned}$$

whence

$$\log(z) - F_{[z_1]}(z) =: C_{[z_1]}$$

is a constant. Thus $\log(z) = C_{[z_1]} + F_{[z_1]}(z)$ is analytic at z_1 ; and since z_1 was arbitrary in $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, we get

$$\log(z) \in \mathcal{A}n(\mathbb{C} \setminus \mathbb{R}_{\leq 0}).$$

In fact, the $C_{[z_1]}$ are all zero. To see this, begin by taking $z_1 = 1$, $w_1 = 0$. Certainly $\log(1) = 0 = F_{[1]}(1)$ from the definitions, and so $C_{[1]} = 0$. Thus on $D(1, 1)$ we have⁷

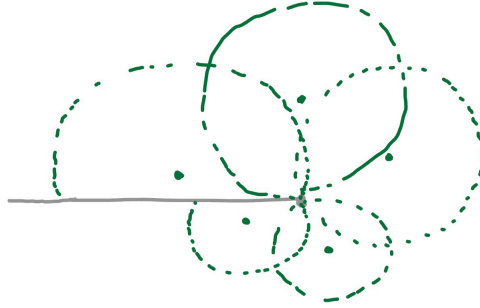
$$e^{\log(z)} = e^{F_{[1]}(z)} = e^{-\sum_{n \geq 1} \frac{(1-z)^n}{n}} = z,$$

so by analytic continuation $e^{\log(z)} = z$ on all of $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. On $D(z_1, |z_1|)$, together with $\frac{z_1}{|z_1|} \in D(1, 1)$, this gives

$$e^{F_{[z_1]}(z) - \log(z)} = e^{w_1} \cdot e^{-\sum_{n \geq 1} \frac{(1 - \frac{z}{z_1})^n}{n}} \cdot \frac{1}{e^{\log(z)}} = z_1 \cdot \frac{z}{z_1} \cdot \frac{1}{z} = 1;$$

in particular, $w_1 - \log(z_1) = F_{[z_1]}(z_1) - \log(z_1) = 2\pi i m$ for some $m \in \mathbb{Z}$. But the imaginary parts of w_1 and $\log(z_1)$ belong to $(-\pi, \pi)$, and so $m = 0$. We conclude that $C_{[z_1]} = 0$ and⁸

$$\log(z) = F_{[z_1]}(z), \quad z \in D(z_1, |z_1|);$$



in other words, $\log(z)$ "stitches together" all the power series centered at different z_1 's.

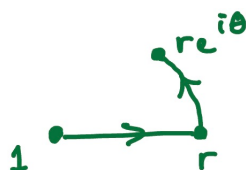
⁷The last equality here holds on $D(1, 1)$ since it holds for $z \in (0, 1) \subset \mathbb{R}$ "by Calculus".

⁸I am cheating a little: if z_1 is in the left half-plane, one should really truncate the disk about it at the real line, as suggested in the picture. The way we have defined it above, $\log(z)$ doesn't extend across the negative reals.

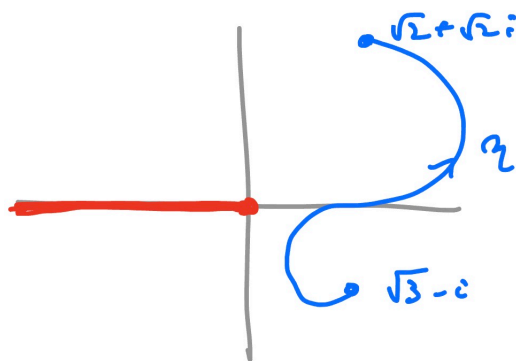
The argument above also exhibits $\log$ as an inverse of $\exp$ (with the power series serving as an intermediate step). A formal consequence is that for $\theta \in (-\pi, \pi)$,

$$\int_1^{re^{i\theta}} \frac{dw}{w} =: \log(re^{i\theta}) = \log(e^{\log r + i\theta}) = \log r + i\theta;$$

alternatively, you can just compute this by direct integration along



II.D.1. EXAMPLE.



Now without a moment's hesitation you can write

$$\begin{aligned} \int_{\eta} \frac{dw}{w} &= \log(w) \Big|_{2e^{-i\frac{\pi}{6}}}^{2e^{i\frac{\pi}{4}}} \\ &= \left(\log 2 + i\frac{\pi}{4} \right) - \left(\log 2 - i\frac{\pi}{6} \right) \\ &= i\frac{5\pi}{12}. \end{aligned}$$

§ Some more general situations.

We explore here a few more functions constructed by integration on subsets of $\mathbb{C}$.

Different branches of $\log(z)$. Suppose that $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected region, and $z_0 \in U$; then

$$\log(z) := w_0 + \int_{z_0}^z \frac{dw}{w}$$

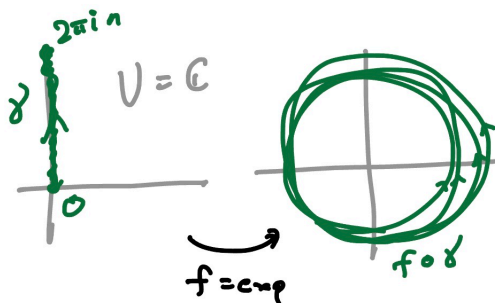
is independent of the path of integration (in U) from z_0 to z . Evaluation at z_0 gives $\log(z_0) = w_0$, and so we should choose w_0 to make $e^{w_0} = z_0$. Adding $2\pi i m$ to w_0 "changes the branch" of $\log$ on U .

Primitive for $f'(z)/f(z)$ on $U \subset \mathbb{C}$. Here we assume that f is a nowhere vanishing holomorphic function on a simply connected region $U \subseteq \mathbb{C}$. We put

$$"\log(f(z))" := w_0 + \int_{z_0}^z \frac{df}{f} = w_0 + \int_{z_0}^z \frac{f'(w)}{f(w)} dw,$$

where $e^{w_0} = f(z_0)$, and the integration again takes place along some path from z_0 to z in U . A warning is in order: this is *not* $f(z)$ plugged into $\log(\cdot)$, or $\int_{f(z_0)}^{f(z)} \frac{dw}{w}$. The reason is that we can't necessarily define $\log$ on $f(U)$, because U simply connected does not imply $f(U)$ simply connected!

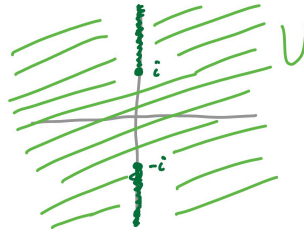
II.D.2. EXAMPLE. $f(z) = \exp(z)$, $U = \mathbb{C}$, $z_0 = 0 = w_0$. The thing called " $\log(f(z))$ " above is a primitive on *all* of $\mathbb{C}$ for $f'(z)/f(z) = e^z/e^z = 1$ which equals 0 at 0; that is, $\log(f(z)) = z$. So for instance for any $n \in \mathbb{Z}$ we have $\log(f(2\pi i n)) = 2\pi i n$,



and we see the notation is deceptive as there is no single branch of log itself that takes all these values!

Other functions.

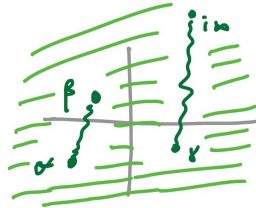
(a) $\frac{1}{1+z^2}$ is holomorphic on $\mathbb{C} \setminus \{\pm i\}$, but this isn't simply connected. To make it so, remove 2 rays:



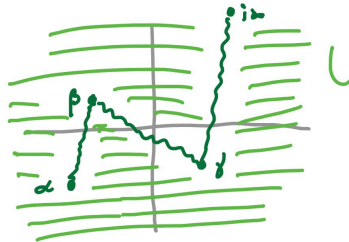
Then

$$\int_0^z \frac{dw}{1+w^2} =: \arctan(z) \in \mathcal{H}ol(U).$$

(b) $f(z) = \frac{1}{\sqrt{(z-\alpha)(z-\beta)(z-\gamma)}}$ is holomorphic and well-defined on the complement of cuts shown:



But this region isn't simply connected. To fix this, remove a strip from β to γ :

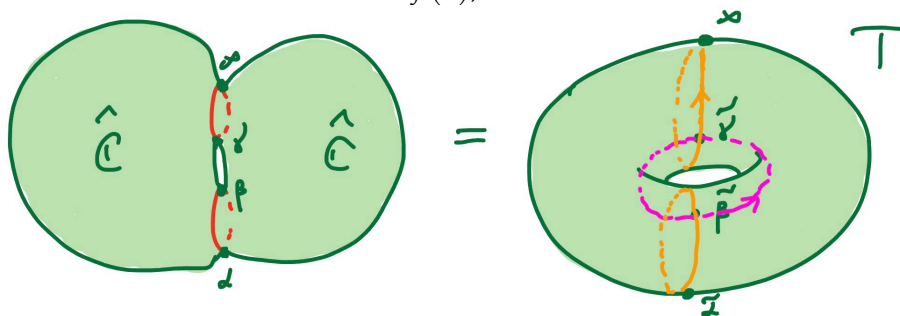


Then the abelian integral

$$A(z) := \int_{z_0}^z \frac{dw}{\sqrt{(w-\alpha)(w-\beta)(w-\gamma)}}$$

is well-defined and holomorphic on U .

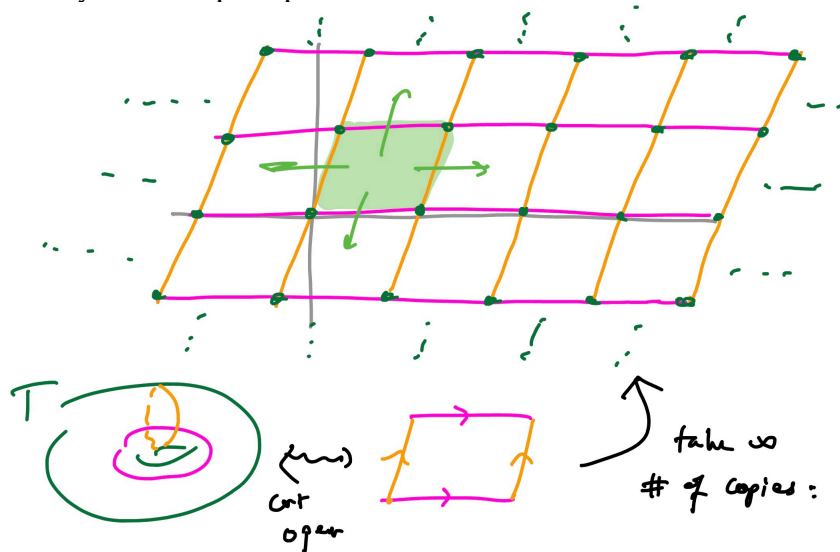
On the Riemann surface of $f(z)$,



this corresponds to cutting open the torus as shown, but one can omit the cuts from $\tilde{\alpha}$ to $\tilde{\beta}$ without sacrificing simple connectivity. This produces a "parallelogram"



corresponding to the fact that the "existence domain of $A(z)$ over T " is actually the complex plane:



The infinitely many translates of the parallelogram fit together to tile $\mathbb{C}$, and are required because A is multivalued "in both horizontal and vertical directions". The dots form a lattice $\Lambda \cong \mathbb{Z}\langle\lambda_1, \lambda_2\rangle \subset \mathbb{C}$, and there is a complex analytic isomorphism from $\mathbb{C}/\Lambda$ to T . We'll see this from a different angle (and more clearly) later on in the course.

§ The dilogarithm.

From $\log(z) \in \mathcal{Hol}(\mathbb{C} \setminus \mathbb{R}_{\leq 0})$ we have $\log(1-z) \in \mathcal{Hol}(\mathbb{C} \setminus \mathbb{R}_{\geq 1})$ and I claim that

$$-\frac{\log(1-z)}{z} \in \mathcal{Hol}(\mathbb{C} \setminus \mathbb{R}_{\geq 1}).$$

As we divided by z , analyticity ($\implies$ holomorphicity) at 0 must be checked:

$$-\log(1-z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots \implies -\frac{\log(1-z)}{z} = 1 + \frac{z}{2} + \frac{z^2}{3} + \cdots$$

Now $\mathbb{C} \setminus \mathbb{R}_{\geq 1}$ is simply connected, and so $-\frac{\log(1-z)}{z}$ has a primitive there. Namely, we define

$$(II.D.3) \quad \text{Li}_2(z) := -\int_0^z \log(1-w) d\log(w) := -\int_0^z \frac{\log(1-w)}{w} dw,$$

where the integral takes place over a path in $\mathbb{C} \setminus \mathbb{R}_{\geq 1}$. This is the **dilogarithm** function, which evidently has the power series representation

$$\text{Li}_2(z) = \sum_{n \geq 1} \frac{z^n}{n^2}$$

on D_1 .

We shall work with the antisymmetrized version⁹

$$\rho(z) := \text{Li}_2(z) - \text{Li}_2(1-z) \in \mathcal{Hol}(\underbrace{\mathbb{C} \setminus (\mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq 0})}_{=:U}),$$

whose differential takes the form

$$\begin{aligned} d(\rho(z)) &= -d \int_0^z \frac{\log(1-w)}{w} dw + d \int_0^{1-z} \frac{\log(1-w)}{w} dw \\ &= -\frac{\log(1-z)}{z} dz - \frac{\log(z)}{1-z} dz \\ &= -\log(1-z) d\log(z) + \log(z) d\log(1-z). \end{aligned}$$

The goal is to establish a property of the dilogarithm which is the analogue of $\log(x) + \log(y) + \log(\frac{1}{xy}) = 0$.

⁹The **Rogers dilogarithm** $R(z) := \text{Li}_2(z) + \frac{1}{2} \log(z) \log(1-z)$ has similar properties to $\rho(z)$.

Consider five points $z_1, \dots, z_5$ such that the “cyclically permuted cross ratios”

$$\begin{aligned}\alpha_1 &= \text{CR}(z_1, z_2, z_3, z_4)^{-1} \\ \alpha_2 &= \text{CR}(z_2, z_3, z_4, z_5)^{-1} \\ \alpha_3 &= \text{CR}(z_3, z_4, z_5, z_1)^{-1} \\ \alpha_4 &= \text{CR}(z_4, z_5, z_1, z_2)^{-1} \\ \alpha_5 &= \text{CR}(z_5, z_1, z_2, z_3)^{-1}\end{aligned}$$

belong to U . (This is true for a “generic” choice of z_i ’s.)

II.D.4. THEOREM (Five-term relation).

$$\sum_{i=1}^5 \rho(\alpha_i) = \frac{\pi^2}{6},$$

independently of the $\{z_i\}$.¹⁰

PROOF. Recall that $\text{CR}(a, b, c, d) := \frac{a-c}{a-d} \cdot \frac{b-d}{b-c}$, so

$$\text{CR}(a, b, c, d)^{-1} = \frac{(a-d)(b-c)}{(a-c)(b-d)}.$$

Step 1: With indices mod 5, I claim that

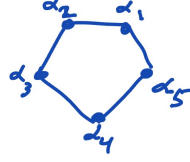
$$(II.D.5) \quad 1 - \alpha_i = \alpha_{i-1} \alpha_{i+1} \quad (\forall i).$$

This is a straightforward calculation:

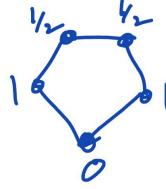
$$\begin{aligned}1 - \alpha_i &= 1 - \frac{z_i - z_{i-2}}{z_i - z_{i+2}} \cdot \frac{z_{i+1} - z_{i+2}}{z_{i+1} - z_{i-2}} \\ &= \frac{(z_i - z_{i+2})(z_{i+1} - z_{i-2}) - (z_i - z_{i-2})(z_{i+1} - z_{i+2})}{(z_i - z_{i+2})(z_{i+1} - z_{i-2})} \\ &= \frac{(z_i - z_{i+1})(z_{i+2} - z_{i-2})}{(z_i - z_{i+2})(z_{i+1} - z_{i-2})} \quad [\text{multiply out, cancel, reorganize}] \\ &= \left(\frac{z_{i-1} - z_{i+2}}{z_{i-1} - z_{i+1}} \cdot \frac{z_i - z_{i+1}}{z_i - z_{i+2}} \right) \left(\frac{z_{i+1} - z_{i-1}}{z_{i+1} - z_{i-2}} \cdot \frac{z_{i+2} - z_{i-2}}{z_{i+2} - z_{i-1}} \right) \\ &= \alpha_{i-1} \cdot \alpha_{i+1}\end{aligned}$$

¹⁰The version for the Rogers dilogarithm is $\sum_i R(\alpha_i) = \frac{\pi^2}{2}$.

Step 2: I like to use the pentagon notation



for a 5-tuple satisfying (II.D.5), for example



Evaluating the antisymmetrized dilogarithm gives

$$\rho(1/2) = 0, \quad \rho(1) = \frac{\pi^2}{6}, \quad \text{and} \quad \rho(0) = -\frac{\pi^2}{6}.$$

Summing around the pentagon gives

$$2\rho(1/2) + 2\rho(1) + \rho(0) = \frac{\pi^2}{6},$$

and the theorem holds for this configuration.

Step 3: It remains to check that as a function of the $\{z_i\}$, the sum $\sum_i \rho(\alpha_i)$ is constant:

$$\begin{aligned} d \sum_{i=1}^5 \rho(\alpha_i) &= \sum_{i=1}^5 d\rho(\alpha_i) \\ &= \sum_{i=1}^5 \log(\alpha_i) d\log(\underbrace{1-\alpha_i}_{\alpha_{i-1}\alpha_{i+1}}) - \sum_{i=1}^5 \log(\underbrace{1-\alpha_i}_{\alpha_{i-1}\alpha_{i+1}}) d\log(\alpha_i) \\ &= \left\{ \sum_i \log(\alpha_i) d\log(\alpha_{i-1}) - \sum_i \log(\alpha_{i+1}) d\log(\alpha_i) \right\} \\ &\quad + \left\{ \sum_i \log(\alpha_i) d\log(\alpha_{i+1}) - \sum_i \log(\alpha_{i-1}) d\log(\alpha_i) \right\} \\ &= 0 + 0 = 0, \end{aligned}$$

where each bracketed term is zero by shifting the index on the right-hand sums. $\square$

Exercises

- (1) Let U be a simply connected open set. Let f be holomorphic and nowhere zero on U . Show that there exists an analytic function g on U such that $g^2 = f$. Does this last assertion remain true if 2 is replaced by an arbitrary positive integer n ?
- (2) Prove a version of Theorem II.D.4 for the Rogers dilogarithm, as suggested above.
- (3) Find a 5-tuple of rational functions of x and y satisfying (II.D.5), the first two of which are x and y .