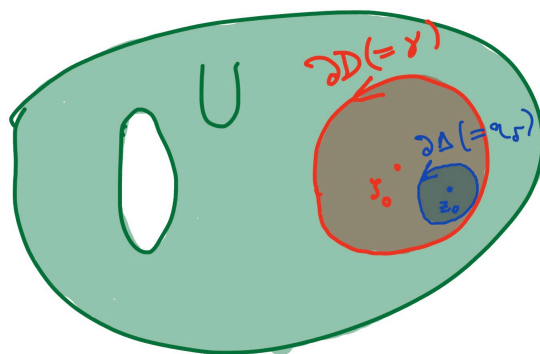


II.E. The Cauchy Integral Formula

§ Reproducing holomorphic functions.

Let U be a region, $D = D(\zeta_0, R)$ a disk with closure $\overline{D} \subset U$. Choose a point $z_0 \in D$, and let $\Delta = D(z_0, \delta)$ be a disk with closure $\overline{\Delta} \subset D$. Write γ and η_δ for the boundaries ∂D and $\partial \Delta$ (traversed counterclockwise as usual) viewed as paths in U .



II.E.1. CLAIM. In $U \setminus \{z_0\}$ we have the homotopy $\gamma \sim \eta_\delta$ of paths.

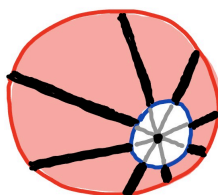
PROOF. Use the parametrizations

$$\gamma(t) = \zeta_0 + Re^{it} \quad \text{and} \quad \eta_\delta(t) = z_0 + \delta \frac{\gamma(t) - z_0}{|\gamma(t) - z_0|}$$

for $t \in [0, 2\pi]$. Then the homotopy is

$$\psi(s, t) := s\gamma(t) + (1 - s)\eta_\delta(t),$$

which maps $[0, 1] \times [0, 2\pi]$ onto the shaded area:



The black lines are traced out by ψ when t is held constant, and following them shows how the homotopy continuously moves one loop to the other. $\square$

II.E.2. THEOREM (Cauchy Integral Formula). *For any $f \in \mathcal{H}ol(U)$, and $z_0 \in D \subset \overline{D} \subset U$ as above,*

$$(II.E.3) \quad \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz = f(z_0).$$

PROOF. Applying the homotopy Cauchy theorem II.C.10 to

$$g(z) := \frac{f(z) - f(z_0)}{z - z_0} \in \mathcal{H}ol(U \setminus \{z_0\})$$

yields

$$\int_{\gamma} g dz = \int_{\eta_\delta} g dz.$$

Since $\lim_{z \rightarrow z_0} g(z) = f'(z_0)$, setting $g(z_0) := f'(z_0)$ makes $g \in \mathcal{C}^0(\overline{D})$. As $\overline{D}$ is compact, there exists $B \in \mathbb{R}_{>0}$ such that $|g(z)| \leq B$ for every $z \in \overline{D} \setminus \{z_0\}$. So

$$\left| \int_{\gamma} g dz \right| = \left| \int_{\eta_\delta} g dz \right| \leq L(\eta_\delta) \|g\|_{\eta_\delta} \leq 2\pi\delta \cdot B \xrightarrow{\delta \rightarrow 0} 0$$

$\implies \int_{\gamma} g dz = 0$, and

$$0 = \int_{\gamma} \frac{f(z) - f(z_0)}{z - z_0} dz = \int_{\gamma} \frac{f(z)}{z - z_0} dz - \int_{\gamma} \frac{f(z_0)}{z - z_0} dz.$$

Applying homotopy Cauchy once more, we have

$$\int_{\gamma} \frac{f(z)}{z - z_0} dz = f(z_0) \int_{\gamma} \frac{dz}{z - z_0} = f(z_0) \int_{\eta_\delta} \frac{dz}{z - z_0} = 2\pi i f(z_0),$$

where the last step is by Proposition II.A.16 or direct integration. $\square$

So we can recover values of a holomorphic function inside a disk from its values on the boundary. Ultimately, this is a consequence of the “rigidity” implied by the Cauchy-Riemann equations.

§ Holomorphic functions are analytic.

As above let $f \in \mathcal{H}\text{ol}(U)$ be a holomorphic function on a region. Fix $z_0 \in U$ and $\overline{D} = \overline{D}(z_0, R) \subset U$ (with interior D), and again write $\gamma = \partial D$. Thinking of $z \in D$ as another fixed point and $w \in \partial D$, the Cauchy integral formula gives

$$\begin{aligned}
 f(z) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} dw \\
 &= \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - z_0) - (z - z_0)} dw \\
 &= \frac{1}{2\pi i} \int_{\gamma} \sum_{n \geq 0} \frac{f(w)(z - z_0)^n}{(w - z_0)^{n+1}} dw \\
 &= \sum_{n \geq 0} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw \right) (z - z_0)^n \\
 \text{(II.E.4)} \quad &=: \sum_{n \geq 0} a_n (z - z_0)^n.
 \end{aligned}$$

Notice that on γ we have $|z - z_0| < |w - z_0|$, which makes the function series converge uniformly there; this justifies interchanging the sum and integral. We could pull $(z - z_0)^n$ outside the integral because it is constant with respect to the integration.

Now thinking of z as a variable, the power series (II.E.4) is convergent on D simply because the computation is valid there. This establishes that its radius of convergence is at least R . The computation also shows that (II.E.4) represents f on D , and thus f is analytic at z_0 . Since $z_0 \in U$ was arbitrary, this yields

II.E.5. THEOREM. $\mathcal{H}\text{ol}(U) = \mathcal{A}\text{n}(U)$.

Of course, we already knew that analytic functions were holomorphic, but it took the Cauchy integral formula to prove the reverse inclusion. Here are two immediate consequences:

II.E.6. COROLLARY. $f \in \mathcal{H}\text{ol}(U) \implies f$ is infinitely complex-differentiable on U .

PROOF. Follows from the Theorem and Cor. I.F.9(a). □

II.E.7. COROLLARY. Let f be holomorphic on $U \subset \mathbb{C}$ and $z_0 \in U$. Set

$r_0 :=$ radius of convergence of the power series for f at z_0 ,

where we remind the reader that this power series is unique. Then

- (a) $r_0 = \max\{r \mid f \text{ admits a holomorphic extension to } D(z_0, r)\}$, and
- (b) $r_0 \geq d(z_0, \partial U)$ (the distance from z_0 to the boundary).

PROOF. The power series is unique by the injectivity of η_{r_0} (see Theorem I.D.23). Certainly (b) follows from (a), and the “ $\leq$ ” part of (a) is immediate. The “ $\geq$ ” part of (a) says that if f extends to a holomorphic function on $D(z_0, r)$ then its power series at z_0 converges there. But applying the argument leading to (II.E.4) to $\overline{D} = \overline{D}(z_0, R) \subset D(z_0, r)$ for any $R < r$, the radius of convergence r_0 is at least R . As this is true for any $R < r$, we get $r_0 \geq r$. $\square$

II.E.8. REMARK. Here is a more explicit check of the lower bound on r_0 in the general context of the last page. Since γ is compact, $|f(w)|$ is bounded by some constant C there; moreover, $|w - z_0| = R$ on γ . This gives the bound

$$\begin{aligned} |a_n| &= \frac{1}{2\pi} \left| \int_{\gamma} \frac{f(w)}{(w - z_0)^{n+1}} dw \right| \leq \frac{1}{2\pi} L(\gamma) \left\| \frac{f(w)}{(w - z_0)^{n+1}} \right\|_{\gamma} \\ &\leq \frac{1}{2\pi} 2\pi R \cdot \frac{C}{R^{n+1}} = \frac{C}{R^n}, \end{aligned}$$

for every $n \geq 0$, whence

$$\frac{1}{r_0} = \limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq \lim_{n \rightarrow \infty} \left(\frac{C}{R^n} \right)^{1/n} = \frac{1}{R} \implies r_0 \geq R.$$

II.E.9. EXAMPLE. Given

$$f(z) = \frac{e^z}{\prod_{i=1}^k (z - \alpha_i)} = \sum_{n \geq 0} A_n (z - z_0)^n,$$

you would compute the $\{A_n\}$ and use the root test to find r_0 . Alternately, just use Corollary II.E.7(a) $\implies r_0 = \min_i |z_0 - \alpha_i|$.

§ Testing for holomorphicity.

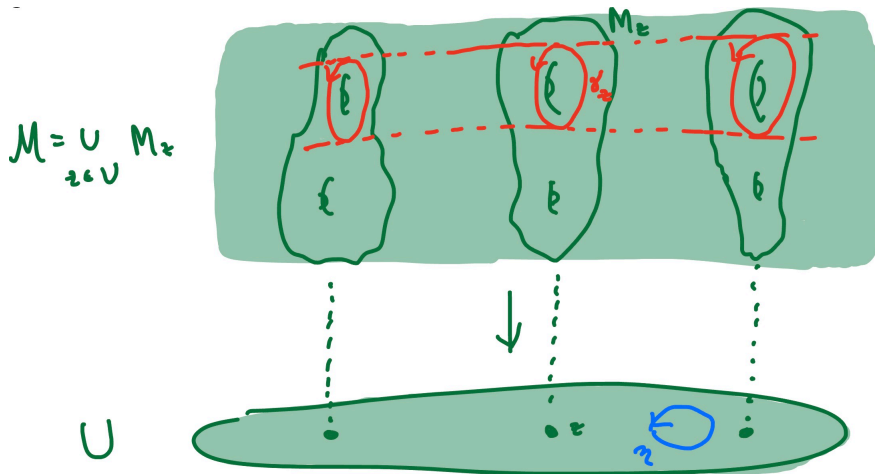
II.E.10. MORERA'S THEOREM. Let U be open, $f \in C^0(U)$, and

$$\int_{\partial R} f dz = 0$$

for all rectangles¹¹ $R \subset U$. Then $f \in \mathcal{H}ol(U)$.

PROOF. By Proposition II.B.4(a), the condition on f guarantees the existence of a local holomorphic primitive F in a neighborhood of every point. But then by Cor. II.E.6, these local primitives are infinitely differentiable, which means in particular that $f = F'$ is differentiable (in a neighborhood of each point and thus in all of U). $\square$

II.E.11. EXAMPLE. In (complex) algebraic geometry, there are integrals over complex manifolds which, by Morera's theorem, are shown to produce holomorphic functions as the complex structure varies. The simplest case is a family of Riemann surfaces (algebraic curves) over an open set $U \subset \mathbb{C}$:



On each M_z , we can compute the **period integral** $f(z) = \int_{\gamma_z} \omega_z$ where ω_z is a family of holomorphic differential 1-forms. Here z is a (possibly local) coordinate pulled back from U ; we also write ω for a 1-form pulling back to ω_z on each M_z .

¹¹Or disks, or triangles, or . . .

If we know that $\omega \wedge dz =: \Omega$ is a holomorphic 2-form on the total space $\mathcal{M}$, then $d\Omega = 0$. Taking any disk $D \subset U$ with boundary $\eta = \partial D$, set

$$\Gamma := \cup_{z \in \eta} \gamma_z = \partial (\cup_{z \in D} \gamma_z) =: \partial \Xi.$$

By Stokes's theorem,

$$\int_{\eta} f dz = \int_{\eta} \left(\int_{\gamma_z} \omega_z \right) dz = \int_{\Gamma} \Omega = \int_{\partial \Xi} \Omega = \int_{\Xi} d\Omega = 0,$$

certifying f as holomorphic by Morera.

§ Generalizations of the integral formula.

Let $\gamma \subset U$ be a $\mathcal{C}^0$ path (not necessarily closed) in an open set, and $g: \gamma \rightarrow \mathbb{C}$ a continuous¹² function defined on (the image of) γ .

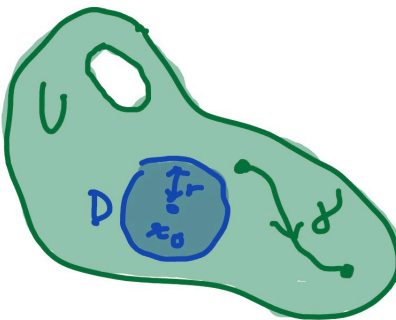
II.E.12. THEOREM-DEFINITION. Define the *Cauchy transform*

$$f(z) := \int_{\gamma} \frac{g(w)}{w - z} dw.$$

Then $f(z) \in \mathcal{Hol}(U \setminus \gamma)$, with derivatives

$$f^{(m)}(z) = m! \int_{\gamma} \frac{g(w)}{(w - z)^{m+1}} dw.$$

PROOF. Note here that we don't have to prove f is equal to anything in particular, so there is no need for the Cauchy integral formula. Pick any $z_0 \in U \setminus \gamma$, and set $r := d(z_0, \partial U \cup \gamma)$.



¹²This condition can even be relaxed.

For all $w \in \gamma$ and $z \in D(z_0, r)$,

$$|z - z_0| < r \leq |w - z_0| \implies \left| \frac{z - z_0}{w - z_0} \right| < 1.$$

Accordingly, developing

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \frac{1}{w - z_0} \sum_{k \geq 0} \left(\frac{z - z_0}{w - z_0} \right)^k \end{aligned}$$

into a power series on γ and multiplying by $g(w)$ gives a uniformly convergent sum, and we may write

$$\begin{aligned} f(z) &:= \int_{\gamma} \frac{g(w)}{w - z} dw = \int_{\gamma} \left(\sum_{k \geq 0} \frac{(z - z_0)^k g(w)}{(w - z_0)^{k+1}} \right) dw \\ &= \sum_{k \geq 0} \int_{\gamma} \frac{(z - z_0)^k g(w)}{(w - z_0)^{k+1}} dw \\ &= \sum_{k \geq 0} \left(\int_{\gamma} \frac{g(w)}{(w - z_0)^{k+1}} dw \right) (z - z_0)^k =: \sum_{k \geq 0} a_k (z - z_0)^k. \end{aligned}$$

Hence $f(z)$ is analytic at z_0 , with

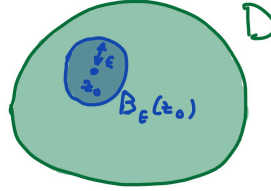
$$f^{(m)}(z_0) = m! a_m = m! \int_{\gamma} \frac{g(w)}{(w - z_0)^{m+1}} dw. \quad \square$$

II.E.13. REMARK. The Cauchy integral formula could *never* reproduce a continuous function which wasn't holomorphic, since by the Theorem and proof above, such integral formulae apparently *always give holomorphic output*. This leads one to ask whether there is a modified Cauchy integral formula for, say, C^∞ functions.

Let $g, h \in C^\infty(U)$ be smooth functions on a region in $\mathbb{C}$, and $\Gamma \subset U$ a compact subset with C^1 boundary. Stokes Theorem II.B.6 says that

$$\begin{aligned} \oint_{\partial\Gamma} (g dz + h d\bar{z}) &= \iint_{\Gamma} d(g dz + h d\bar{z}) = \iint_{\Gamma} \left(\frac{\partial g}{\partial \bar{z}} d\bar{z} \wedge dz + \frac{\partial h}{\partial z} dz \wedge d\bar{z} \right) \\ &= \iint_{\Gamma} \left(\frac{\partial h}{\partial z} - \frac{\partial g}{\partial \bar{z}} \right) dz \wedge d\bar{z}. \end{aligned}$$

Applying this¹³ in the setting where $\Gamma = \overline{D}$ is a disk with $z_0 \in D$, $g(z) = \frac{f(z)}{z-z_0}$ ($f \in C^\infty(U)$), and $h(z) = 0$, we get

$$\begin{aligned} \iint_D \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z-z_0} &:= \lim_{\epsilon \rightarrow 0} \iint_{D \setminus B_\epsilon(z_0)} \frac{\partial f}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z-z_0} \\ &= - \lim_{\epsilon \rightarrow 0} \oint_{\partial(D \setminus B_\epsilon(z_0))} \frac{f(z)}{z-z_0} dz \end{aligned}$$


using the fact that $\frac{\partial}{\partial \bar{z}}$ of $\frac{1}{z-z_0}$ is zero. Writing $C_\epsilon(z_0) = \partial B_\epsilon(z_0)$, this

$$\begin{aligned} &= \lim_{\epsilon \rightarrow 0} \oint_{C_\epsilon(z_0)} \frac{f(z)}{z-z_0} dz - \oint_{\partial D} \frac{f(z)}{z-z_0} dz \\ &= 2\pi i f(z_0) - \oint_{\partial D} \frac{f(z)}{z-z_0} dz. \end{aligned}$$

The last equality here comes from the bound

$$\begin{aligned} \left| \int_{C_\epsilon} \frac{f(z) - f(z_0)}{z-z_0} dz \right| &\leq 2\pi\epsilon \left\| \frac{f(z) - f(z_0)}{z-z_0} \right\|_{C_\epsilon} \\ &\leq 2\pi \|f(z) - f(z_0)\|_{C_\epsilon} \rightarrow 0 \quad \text{since } f \text{ is } C^\infty. \end{aligned}$$

Finally, using

$$\begin{aligned} dz \wedge d\bar{z} &= (dx + i dy) \wedge (dx - i dy) = -2i dx \wedge dy = -2i dA \\ \implies \frac{1}{2\pi i} dz \wedge d\bar{z} &= \frac{-2i}{2\pi i} dA = -\frac{1}{\pi} dA, \end{aligned}$$

we may rewrite the above in the form

$$(II.E.14) \quad f(z_0) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(z)}{z-z_0} dz - \frac{1}{\pi} \iint_D \frac{\partial f}{\partial \bar{z}} \frac{dA}{z-z_0}$$

This is the **generalized Cauchy integral formula**. Notice that we need to introduce a term involving $\partial f / \partial \bar{z}$ in order to be able to reproduce a non-holomorphic function.

¹³Technically, we are applying it not on U but on $U \setminus B_\epsilon(z_0)$. Also, I will not distinguish between the integral over $\overline{D}$ and the integral over D .

§ The winding number.

A special case of Theorem II.E.12 is where $g \equiv 1$ and γ is *closed*. Here we shall assume that $\gamma: [a, b] \rightarrow U$ is at least piecewise $\mathcal{C}^1$, so that we may write

$$(II.E.15) \quad f(z) := \int_{\gamma} \frac{1}{w-z} dw = \int_a^b \frac{\gamma'(t)}{\gamma(t)-z} dt.$$

Now view z for the moment as *fixed*, and consider the related function

$$F(s) := \int_a^s \frac{\gamma'(t)}{\gamma(t)-z} dt, \quad s \in [a, b].$$

This is differentiable (with the obvious derivative) except at finitely many points, and we can write

$$\begin{aligned} \frac{d}{ds} \left[\overbrace{e^{-F(s)}(\gamma(s)-z)}^{\tilde{F}(s):=} \right] &= e^{-F(s)}\gamma'(s) - F'(s)e^{-F(s)}(\gamma(s)-z) \\ &= e^{-F(s)}(\gamma(s)-z) \underbrace{\left(\frac{\gamma'(s)}{\gamma(s)-z} - F'(s) \right)}_0 = 0. \end{aligned}$$

So $\tilde{F}$ is on the one hand continuous, and on the other constant except (possibly) at finitely many points. Thus it is constant (on $[a, b]$).

That is, we have¹⁴

$$\begin{aligned} e^{-F(s)}(\gamma(s)-z) &= C (\neq 0) \\ \implies \gamma(s)-z &= Ce^{F(s)} \\ \implies Ce^{F(b)} = \gamma(b)-z &= \gamma(a)-z = Ce^{F(a)} \\ \implies e^{F(b)} = e^{F(a)} &= e^0 = 1 \\ \implies f(z) = F(b) &= 2\pi i k, \text{ for some } k \in \mathbb{Z} \\ \implies \frac{1}{2\pi i} f(z) &\in \mathbb{Z}. \end{aligned}$$

By Theorem II.E.12, $f(z) \in \mathcal{H}ol(U \setminus \gamma)$.

¹⁴The constant C here is nonzero because $z \notin \gamma$ and $0 \notin \text{range of exp}$. Also, $\gamma(a) = \gamma(b)$ since γ is closed.

But holomorphic functions are continuous, and a $2\pi i\mathbb{Z}$ -valued continuous function is *constant on connected components*. This brings us to the

II.E.16. DEFINITION. Let $U \subset \mathbb{C}$ be open, $\gamma \subset U$ be a closed piecewise $\mathcal{C}^1$ path, and $z \in U \setminus \gamma$. The **winding number** of γ with respect to z is

$$(II.E.17) \quad W(\gamma, z) := \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - z} dw.$$

The argument above has proved the following:

II.E.18. PROPOSITION. $W(\gamma, z)$ is integer-valued, and constant on connected components of $U \setminus \gamma$.

II.E.19. EXAMPLE. $U = \mathbb{C}, z = 0; \gamma(t) = e^{2\pi i n t}, t \in [0, 1] \implies$

$$W(\gamma, 0) = \frac{1}{2\pi i} \int_0^1 \frac{2\pi i n e^{2\pi i n t}}{e^{2\pi i n t}} dt = n,$$

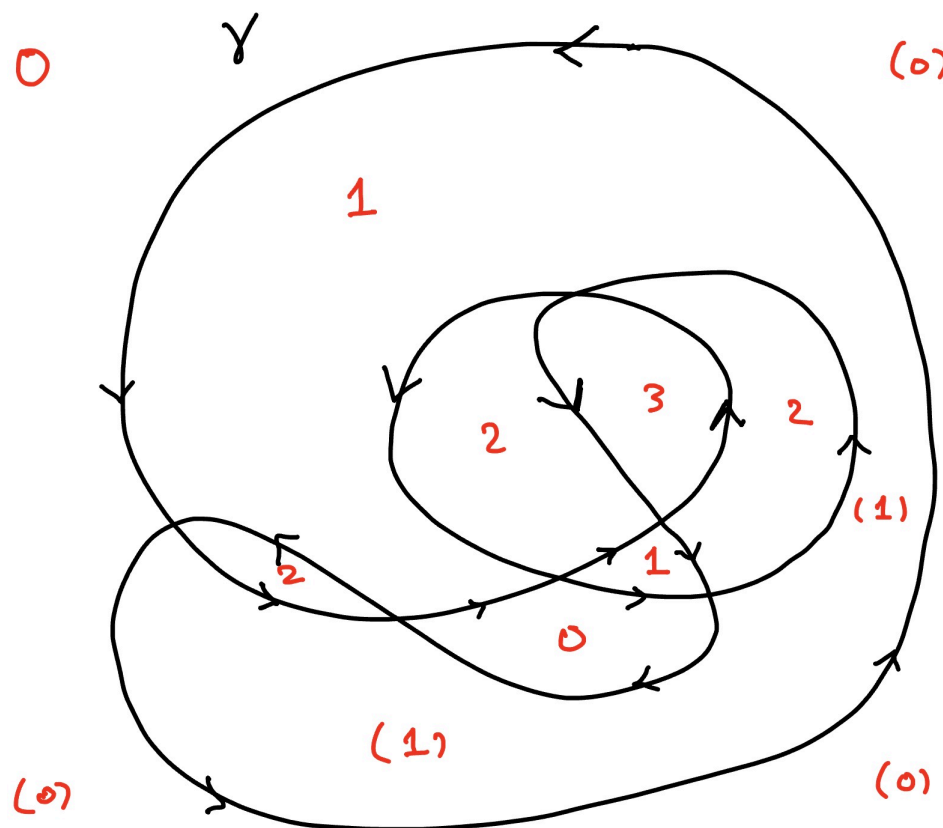
using (II.E.15). This is just the case of a circular loop winding around the origin n times.

II.E.20. EXAMPLE. What can we say in general when $U = \mathbb{C}$? Since γ compact $\implies \gamma$ bounded, there exists $R > 0$ sufficiently large that $\gamma \subset D(0, R)$. We can also assume (since $L(\gamma) < \infty$) that $R > \frac{L(\gamma)}{2\pi}$. Now let $z := 2R$ and write

$$|W(\gamma, 2R)| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{1}{w - 2R} dw \right| \leq \frac{L(\gamma)}{2\pi} \left\| \frac{1}{w - 2R} \right\|_{\gamma} < \frac{L(\gamma)}{2\pi} \frac{1}{R} < 1.$$

Since $W(\gamma, 2R) \in \mathbb{Z}$, it must be 0. By the Proposition, it is 0 on all of $(\mathbb{C} \setminus \gamma)^{\text{ub}}$, the (unbounded) connected component of $\mathbb{C} \setminus \gamma$ containing $D(0, R)^c$ (and in particular $2R$).

II.E.21. EXAMPLE. Here are the values of $W(\gamma, z)$ on connected components of $\mathbb{C} \setminus \gamma$ for a somewhat general choice of γ :



Together with the “strong” version of local Cauchy, the above leads to a slightly different proof of the Cauchy integral formula. (Note that the last Theorem, on which the material in this section is based, did not use Cauchy’s formula.) Moreover, we get a statement which permits more general paths:

II.E.22. THEOREM (Cauchy Integral Formula, v. 2.0). *Let D be an open disk, $f \in \mathcal{H}ol(D)$, $\gamma \subset D$ a closed C^1 path,¹⁵ and $z_0 \in D \setminus \gamma$. Then*

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - z_0} dz = W(\gamma, z_0) \cdot f(z_0).$$

¹⁵The result actually holds without the C^1 assumption.

PROOF. Write

$$g(z) := \frac{f(z) - f(z_0)}{z - z_0}.$$

Since f is continuous, we have

$$(II.E.23) \quad \lim_{z \rightarrow z_0} (z - z_0)g(z) = 0.$$

By the “strong” local Cauchy theorem II.C.3, since $z_0 \notin \gamma$, and $g \in \mathcal{Hol}(D \setminus \{z_0\})$ satisfies (II.E.23),

$$\begin{aligned} 0 &= \frac{1}{2\pi i} \oint_{\gamma} g(z) dz \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz - \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z_0)}{z - z_0} dz \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz - f(z_0) \cdot \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} \\ &= \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0} dz - f(z_0) \cdot W(\gamma, z_0) \end{aligned}$$

which proves the result. $\square$

§ Entire functions.

An **entire function** is just an element of $\mathcal{Hol}(\mathbb{C})$.

II.E.24. LIOUVILLE’S THEOREM. *If f entire and bounded¹⁶ then f is constant.*

II.E.25. COROLLARY. *If a function is entire and nonconstant, then it is unbounded.*

II.E.26. EXAMPLES. $\cos, \sin, \exp$.

II.E.27. COROLLARY (Fundamental Theorem of Algebra). *Any non-constant polynomial*

$$P(z) = a_n z^n + \cdots + a_1 z + a_0 \quad (n > 0, a_n \neq 0)$$

has a root in $\mathbb{C}$.

¹⁶i.e. there exists $C \in \mathbb{R}_{\geq 0}$ such that $|f(z)| \leq C$ ($\forall z \in \mathbb{C}$)

PROOF OF COR. II.E.27. Assume P has no root; then $\frac{1}{P}$ is entire. Writing

$$P(z) = a_n z^n \left(1 + \frac{b_1}{z} + \cdots + \frac{b_n}{z^n} \right),$$

we find that for any $M > 0$, $\exists R \in \mathbb{R}_+$ such that

$$|z| > R \implies |P(z)| > M.$$

Hence $\frac{1}{P} \rightarrow 0$ as $|z| \rightarrow \infty$, and defining $(\frac{1}{P})(\infty) := 0$ therefore makes $\frac{1}{P}$ continuous on $\hat{\mathbb{C}}$. By compactness of $\hat{\mathbb{C}}$, $\frac{1}{P}$ is bounded; and so Liouville $\implies \frac{1}{P}$ is constant $\implies P$ constant. ✖ $\square$

PROOF OF LIOUVILLE. By Cor. II.E.7, f entire $\implies$ the power series $\sum a_k z^k$ for f at 0 converges on all of $\mathbb{C}$ ($r_0 = \infty$). So if all $a_1, a_2, \dots = 0$ in this expansion, then $f \equiv a_0$ is constant.

Let $\gamma_R = \partial D_R$, $R > 0$ arbitrary. Recall that the Cauchy integral formula yields a power series expansion about any point in D_R ; in particular, about 0, we have

$$f(z) = \sum_{k \geq 0} \left(\frac{1}{2\pi i} \int_{\gamma_R} \frac{f(w)dw}{w^{k+1}} \right) z^k$$

Then

$$\begin{aligned} |a_k| &= \frac{1}{2\pi} \left| \int_{\gamma_R} \frac{f(w)dw}{w^{k+1}} \right| \\ &\leq \frac{1}{2\pi} \cdot L(\gamma_R) \cdot \left\| \frac{f(w)}{w^{k+1}} \right\|_{\gamma_R} \\ &\leq \frac{1}{2\pi} \cdot 2\pi R \cdot \frac{C}{R^{k+1}} \\ &= \frac{C}{R^k} \xrightarrow{R \rightarrow \infty} 0 \text{ for } k \geq 1. \end{aligned}$$

$\square$

A generalization of Liouville is the

II.E.28. PROPOSITION. *If f is entire with polynomial bound*

$$|f(z)| \leq C|z|^m \text{ for } |z| \geq R_0 > 0,$$

then f is a polynomial of degree $\leq m$.

IDEA OF PROOF FOR $m = 1$. We are given that $|f(z)| \leq C|z|$ for $|z| \geq R_0$. So

$$|a_k| \leq \frac{1}{2\pi} \cdot 2\pi R \left\| \frac{f(w)}{w^{k+1}} \right\|_{\gamma_R} \leq R \cdot \frac{C \cdot R}{R^{k+1}} = \frac{C}{R^{k-1}} \xrightarrow{R \rightarrow \infty} 0$$

for $R \geq R_0$ and $k \geq 2$. $\square$

The case $m > 1$ is an exercise.

II.E.29. REMARK. If you want to get rid of “for $|z| \geq R_0$ ”, then make the bound instead $|f(z)| \leq C_0 + C_1|z|^m$ ($\forall z \in \mathbb{C}$). This implies the above bound with $C = 1 + C_1$, $R_0 = C_0^{1/m}$, since

$$|z| \geq C_0^{1/m} \implies |z|^m \geq C_0 \implies (1 + C_1)|z|^m \geq C_0 + C_1|z|^m.$$

Exercises

- (1) Compute (a) $\int_{|z|=1} e^z z^{-n} dz$, (b) $\int_{|z|=2} \frac{dz}{z^2+1}$, and (c) $\int_{|z|=1} \frac{\cos(z^2)}{z} dz$. You may use the local Cauchy integral formula (Theorem II.E.2), the formula for derivatives in Theorem II.E.12, and partial fractions — nothing on residues.
- (2) Compute $\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$ under the condition $|a| \neq \rho$. [Hint: make use of the equations $z\bar{z} = \rho^2$ and $|dz| = i\rho \frac{dz}{z}$, as well as the Cauchy integral formula and partial fractions; again, no residue theorem, as we don't yet have it.]
- (3) (a) Show that the association $f \mapsto \frac{f'}{f}$ (where f is holomorphic) sends products to sums.
 (b) If $P(z) = (z - a_1) \cdots (z - a_n)$, where $a_1, \dots, a_n$ are the roots, what is P'/P ?
 (c) Let γ be a closed path such that none of the roots of P lie on γ . Show that $\frac{1}{2\pi i} \int_{\gamma} \frac{P'(z)}{P(z)} dz$ equals $W(\gamma, a_1) + \cdots + W(\gamma, a_n)$.
- (4) Prove that for any simple closed C^1 path γ , there exists $\alpha \in \mathbb{C}$ such that $W(\gamma, \alpha) \neq 0$. [See problem #3 on p. 118 of Ahlfors.]
- (5) Prove the generalization of Liouville's theorem in Proposition II.E.28.

- (6) If $f(z)$ is holomorphic for $|z| < 1$ and $|f(z)| \leq \frac{1}{1-|z|}$ there, find the best estimate of $|f^{(n)}(0)|$ that [the inequalities derived from] Cauchy's theorem will yield.