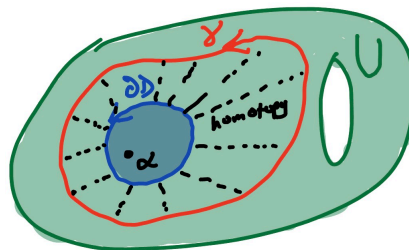


II.F. Cauchy's Theorem (part 2)

§ Homotopy classes of paths.

The point of Cauchy's integral formula is to compute integrals, and we want a stronger version than the "homotopy form". Let's recall what we know: given

- a region (connected open set) U ,
- a closed disk $\overline{D} \subset U$,
- a point $\alpha \in D$ in the open disk,
- a function $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$, and
- a closed path $\gamma \sim \partial D$ in $U \setminus \{\alpha\}$,



the Cauchy integral formula and homotopy Cauchy theorem give

$$(II.F.1) \quad f(\alpha) \stackrel{(II.E.2)}{=} \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - \alpha} dz \stackrel{(II.C.10)}{=} \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z - \alpha} dz$$

More generally if $\gamma \sim \eta$ in $U \setminus \{\alpha\}$, with $\eta \subset D \setminus \{\alpha\}$ and winding number $W(\eta, \alpha) = n$, then

$$(II.F.2) \quad n \cdot f(\alpha) = \frac{1}{2\pi i} \int_{\eta} \frac{f(z)}{z - \alpha} dz$$

by replacing (II.C.10) with (II.E.22).

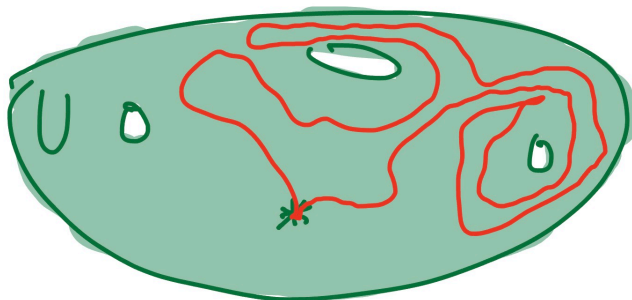
In either case, the argument requires having the path γ of integration to be *homotopic* in $U \setminus \{\alpha\}$ to a path in a disk containing α and contained in U . If this is not true for a given γ , might something like (II.F.1) or (II.F.2) hold anyway? We are heading toward a *homology* version of Cauchy, which is stronger because *it takes far less for closed paths to be homologous than for them to be homotopic*. So let's review these competing "global topological" notions.

II.F.3. DEFINITION. The **first homotopy group** or **fundamental group** of U has the underlying set

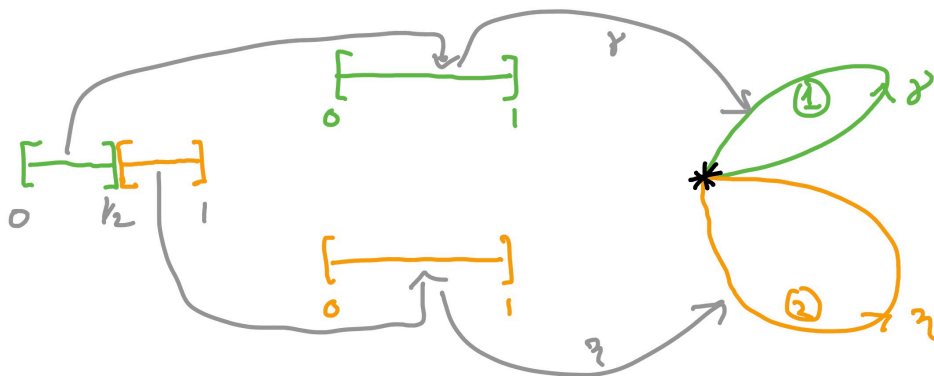
$$(II.F.4) \quad \pi_1(U, *) := \frac{\{\text{closed } \mathcal{C}^0 \text{ paths starting \& ending at } *\}}{\text{homotopy equivalence } \sim}$$

where the base point "*" is some fixed point of U . The denominator of (II.F.4) means: γ and η are the same element of π_1 iff $\gamma \sim \eta$ in U .

A path is the "trivial element" $\{*\}$ in π_1 iff $\gamma \sim \{*\}$: i.e., there exists a $\mathcal{C}^0$ map $\psi : [0, 1] \times [0, 1] \rightarrow U$ with $\psi_0(t) = \gamma(t)$ and $\psi_1(t) = \{*\}$ ($\forall t$). Intuitively, this means (viewing γ as a "rope") you can stand at $\{*\}$, grab two ends of the rope and pull to yourself without the rope passing through any holes in U .



Given $\gamma, \eta : [0, 1] \rightarrow U$, define $\eta \cdot \gamma : [0, 1] \rightarrow U$ by concatenation:



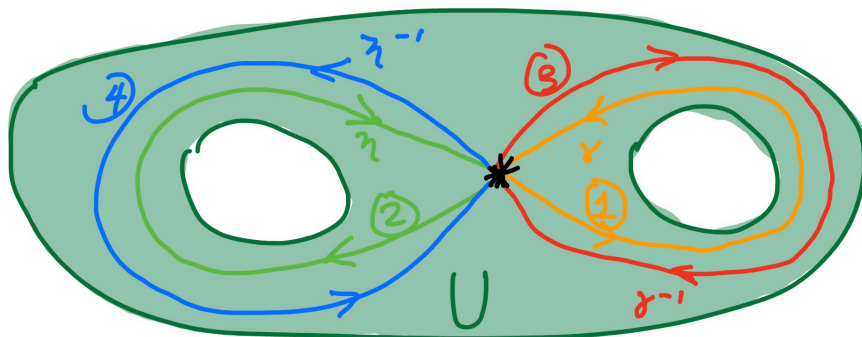
This makes π_1 into a group, with identity $\{*\}$ (constant path) and inverse " γ^{-1} " just γ traversed backwards. (Why does this work?)

II.F.5. REMARK. The homotopy group is usually nonabelian. Of course, for a simply connected set everything is homotopic to $\{*\}$ and π_1 is trivial; for an annulus or the complement of a point in a disk, $\pi_1 \cong \mathbb{Z}$ (why?). But in general, we'll have $\gamma \cdot \eta \not\sim \eta \cdot \gamma$, or equivalently

$$\eta^{-1} \cdot \gamma^{-1} \cdot \eta \cdot \gamma \not\sim \{*\}$$

for the **commutator** of η and γ . Crucial here is that we aren't allowed to subdivide the path and cancel pieces. Imagine tying the ends of

four strings together, holding the beginning of γ in one hand and the end of η^{-1} in the other, and trying to pull the whole thing towards you. One can actually try this, replacing the holes in U in the picture below by the legs of a table (or friend), but don't be reckless.



In any case, it won't work,¹⁷ because this commutator is nontrivial.

Accordingly, homotopy Cauchy can't tell us anything about the integral over $\eta^{-1} \cdot \gamma^{-1} \cdot \eta \cdot \gamma \dots$

§ Homology classes of chains.

... which is, of course, completely ridiculous: if you integrate over $\eta^{-1} \cdot \gamma^{-1} \cdot \eta \cdot \gamma$ (regardless of the integrand $f \in \mathcal{H}ol(U)$), you get zero by cancellation!! So the "fundamental" group is missing something rather fundamental.

II.F.6. DEFINITION. Let $\gamma \subset U$ be a closed path. We say that γ is **homologous to zero** on U , written $\gamma \equiv_{\text{hom}} 0$, if¹⁸

$$0 = W(\gamma, \alpha) := \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - \alpha} \quad (\forall \alpha \in U^c).$$

Thus for the reason we just described,

$$\eta^{-1} \cdot \gamma^{-1} \cdot \eta \cdot \gamma \equiv_{\text{hom}} 0.$$

¹⁷Unless you break the legs of your table or friend.

¹⁸Recall that $U^c := \mathbb{C} \setminus U$. The definition says that the winding number of γ about points *not* in U is zero.

The reason I wrote “chains” (in the section title) is that I want you to think of homology in terms of *subdividable* objects, in contrast to homotopy. Why? Because integrals can be subdivided *by definition* (Riemann), and homology is defined¹⁹ in terms of winding numbers (which are integrals).

II.F.7. DEFINITION. On a region $U \subset \mathbb{C}$, we define the following:

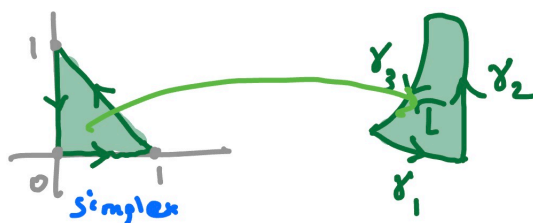
(i) A **0-chain** is a formal sum of points (in U) with integer coefficients, viz. $\sum m_i \cdot p_i$.

(ii) A **1-chain** is a formal sum of paths (in U) with integer coefficients, viz. $\sum m_i \cdot \gamma_i$.



- There is a notion of “boundary” which sends 1-chains to 0-chains, e.g. $\partial\gamma_2 = p_2 - p_1$ and $\partial(\gamma_1 + \gamma_2) = p_3 - p_1$ in the picture.
- Each “piece” is parametrized but (except for the direction) we forget this.

(iii) A **2-chain** is a formal sum of curvy triangles (in U) with integer coefficients, viz. $\sum m_i \tau_i$.



- The boundary of a 2-chain consists of the images of the boundary components of the simplex: $\partial\tau = \gamma_1 + \gamma_2 + \gamma_3$.
- We can present any closed region as a 2-chain by subdividing it into curvy triangles (triangulating).

¹⁹in our *ad hoc* definition above. There is a better, more general definition, to be given in a moment.

(iv) We write sums of chains with “+”, not “·”, whether or not they are end-to-end; they *commute by definition*. Also,



are identified (and similarly for 2-chains).

(v) A 1-chain γ is **closed** $\iff \partial\gamma = 0$. Define the **first homology group** of U by

$$H_1(U) := \frac{\{\text{closed 1-chains in } U\}}{\partial\{2\text{-chains in } U\}} = \frac{\text{“cycles”}}{\text{“boundaries”}}$$

The element in H_1 determined by γ is called the **homology class** of γ and is denoted $[\gamma]$.

Cauchy’s theorem will just say that integrals of holomorphic functions on U are *well-defined on homology classes*; that is,

$$\text{“} \int_{[\gamma]} f dz \text{”}$$

makes sense.

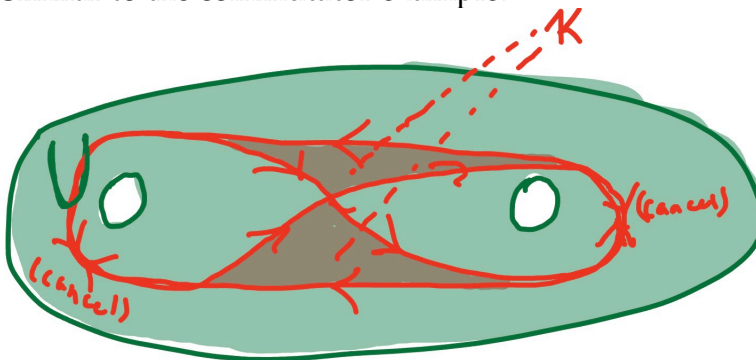
The assertion in the background here is that the two definitions of “homologous to zero” are the same: for a closed 1-chain γ on U ,

$$(II.F.8) \quad \left\{ \begin{array}{l} \gamma \equiv 0 \\ \text{hom} \\ \text{(in } U) \end{array} \right\} \iff \left\{ \begin{array}{l} W(\gamma, \alpha) = 0 \\ (\forall \alpha \in U^c) \end{array} \right\} \stackrel{(+)}{\iff} \left\{ \begin{array}{l} \gamma = \partial\mathcal{K}, \\ \mathcal{K} \text{ a 2-chain in } U \end{array} \right.$$

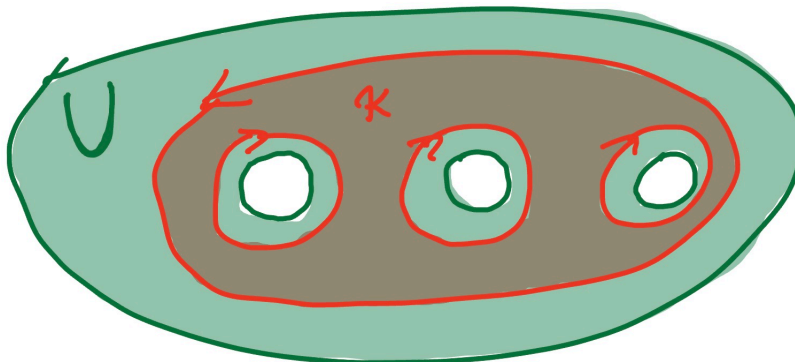
We will prove (+) below. Clearly, the right-hand criterion is better for drawing pictures:

II.F.9. EXAMPLES (of γ homologous to 0).

(1) Similar to the commutator example:



(2) The big loop is homologous to the sum of the three small ones:



Now if $\alpha \notin U$, then $\frac{1}{z-\alpha} \in \mathcal{H}ol(U)$; thus by homotopy Cauchy,

$$\begin{aligned} \gamma \sim \{*\} \text{ (in } U) &\implies \int_{\gamma} \frac{dz}{z-\alpha} = \int_{\{*\}} \frac{dz}{z-\alpha} = 0 \quad (\forall \alpha \in U^c) \\ &\implies W(\gamma, \alpha) = 0 \quad (\forall \alpha \in U^c) \\ &\implies \gamma \equiv 0 \text{ (in } U)_{\text{hom}} \\ &\implies [\gamma] = 0 \text{ in } H_1(U). \end{aligned}$$

So: “homotopic to 0 $\implies$ homologous to 0”, but *not* vice versa (think commutators). We formalize this in the

II.F.10. DEFINITION. The **Hurewicz homomorphism** is the natural map from the fundamental group to first homology given by

$$\begin{aligned} \pi_1(U) &\rightarrow H_1(U) \\ \gamma &\mapsto [\gamma]. \end{aligned}$$

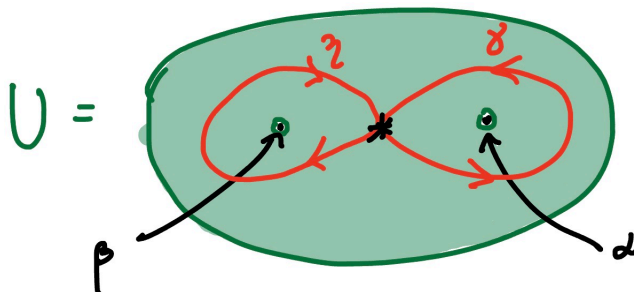
It sends $\eta \cdot \gamma \mapsto [\eta] + [\gamma]$ and $\eta^{-1}\gamma^{-1}\eta\gamma \mapsto 0$.

Clearly this homomorphism is surjective. In fact, it is true that

$$(II.F.11) \quad H_1(U) \cong \pi_1(U) / [\pi_1(U), \pi_1(U)],$$

where $[\pi_1, \pi_1]$ is the *commutator subgroup* generated by all elements of the form $\eta^{-1}\gamma^{-1}\eta\gamma$.

II.F.12. EXAMPLE. Consider the elliptical region minus two points:



Here $\pi_1(U, *)$ is the free group $\mathfrak{F}\{\gamma, \eta\}$ on the two generators shown. Its elements are the *words* $\omega = \gamma^{a_1} \eta^{b_1} \gamma^{a_2} \eta^{b_2} \dots \eta^{b_n}$. The first homology $H_1(U) \cong \mathbb{Z} \oplus \mathbb{Z}$ is the free *abelian* group on $[\gamma]$ and $[\eta]$. The Hurewicz map sends

$$\omega \mapsto [\omega] = (\sum_{i=1}^n a_i)[\gamma] + (\sum_{i=1}^n b_i)[\eta] =: a[\gamma] + b[\eta].$$

Notice that $a = W(\omega, \alpha)$ and $b = W(\omega, \beta)$.

Before explaining (II.F.8) above, I recall properties of the winding number:

- (i) It's an integral. Regardless of the integrand,²⁰ you can count on having $\int_{\gamma+\eta} = \int_{\gamma} + \int_{\eta}$. (This is why homology works so well with integration.) So

$$W(\gamma, \alpha) + W(\eta, \alpha) = W(\gamma + \eta, \alpha)$$

for closed η, γ .

- (ii) It's an integer, and it is *constant as α varies in a connected component of $\mathbb{C} \setminus |\gamma|$* .

PROOF OF (II.F.8). Of course, what we are proving is $(\dagger)$.

Step 1 $\gamma = \partial K \implies W(\gamma, \alpha) = 0$ for any $\alpha \notin U$: Begin by writing our boundary 1-chain as

$$\gamma = \sum m_i \cdot \partial \tau_i,$$

²⁰Hmmm, well, there is a notion of *iterated integrals*, which are only well-defined on homotopy classes. But we won't go there in this class.

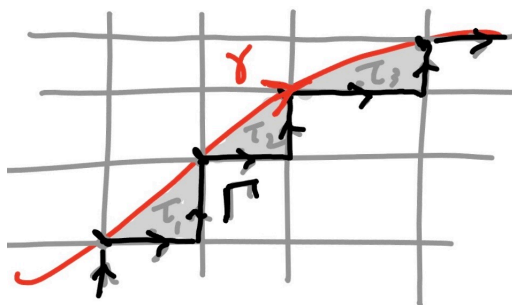
$\tau_i \subset U$. By triangulating, we can make the $\{\tau_i\}$ as small as needed; in particular, we can make them fit in disks of radii $< d(\gamma, U^c)$. Since these disks are simply connected, the $\partial\tau_i$ are homotopic to 0. By homotopy Cauchy, we then have

$$\int_{\gamma} f dz = \sum m_i \int_{\partial\tau_i} f dz = 0$$

for $f \in \mathcal{Hol}(U)$, and of course $\frac{1}{z-\alpha} \in \mathcal{Hol}(U)$ when $\alpha \notin U$.

Step 2 $W(\gamma, \alpha) = 0$ ($\forall \alpha \notin U$) $\implies \gamma = \partial K$: This is the harder direction, as we must construct K .

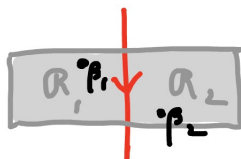
Partition γ so that points are closer together than $d(\gamma, U^c)$. Then “rectangularize” the path to obtain Γ .



Clearly $\Gamma - \gamma = \partial(\sum \tau_i)$ and Γ, γ close together²¹ $\implies \Gamma, \gamma$ homotopic $\implies \Gamma, \gamma$ have the same winding numbers about $\alpha \notin U$. So we can replace γ by Γ .

Now subdivide the plane into rectangles $\overline{\mathcal{R}}_i$, using horizontal and vertical extensions of Γ , and let $\beta_i \in \mathcal{R}_i$ be arbitrary.

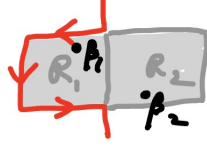
II.F.13. LEMMA. Let η be a rectangular path, with segment between $\mathcal{R}_1$ and $\mathcal{R}_2$:



Then $W(\eta, \beta_1) \neq W(\eta, \beta_2)$. [We remark here that the path could have some multiplicity along this segment, but we are assuming this multiplicity is 1.]

²¹In the sense of Definition II.C.7.

PROOF OF LEMMA. "Divert" the path by adding $\partial\mathcal{R}_1$:



Then β_1 and β_2 belong to the same connected component of $\mathbb{C} \setminus |\eta + \partial\mathcal{R}_1|$. So

$$\begin{aligned} 0 &= W(\eta + \partial\mathcal{R}_1, \beta_1) - W(\eta + \partial\mathcal{R}_1, \beta_2) \\ &= (W(\eta, \beta_1) - W(\eta, \beta_2)) + (W(\partial\mathcal{R}_1, \beta_1) - W(\partial\mathcal{R}_1, \beta_2)) \\ &= \{W(\eta, \beta_1) - W(\eta, \beta_2)\} + (1 - 0) \end{aligned}$$

forces the term in braces to be nonzero as desired. $\square$

Now set $m_i := W(\Gamma, \beta_i)$, $\beta_i \in \mathcal{R}_i$ arbitrary. By hypothesis, we have $\alpha \notin U \implies W(\Gamma, \alpha) = 0$; so for each $m_i \neq 0$ it follows that $\beta_i \in U$, and thus that $\overline{\mathcal{R}_i} \subset U$. Therefore

$$\mathcal{K} := \sum m_j \overline{\mathcal{R}_j}$$

is 2-chain²² in U . Notice that for $\beta_i \in \mathcal{R}_i$,

$$(II.F.14) \quad W(\partial\mathcal{K}, \beta_i) = \sum m_j W(\partial\mathcal{R}_j, \beta_i) = m_i = W(\Gamma, \beta_i).$$

So I claim that $\Gamma = \partial\mathcal{K}$. Suppose otherwise: then the closed rectangular 1-chain $\Gamma - \partial\mathcal{K}$ is nonzero, hence has a segment. That segment divides 2 rectangles (say, $\mathcal{R}_1$ and $\mathcal{R}_2$), but by (II.F.14)

$$W(\Gamma - \partial\mathcal{K}, \beta_1) = W(\Gamma, \beta_1) - W(\partial\mathcal{K}, \beta_1) = 0 = W(\Gamma - \partial\mathcal{K}, \beta_2),$$

contradicting the Lemma. We conclude that Γ (hence γ) is a boundary. This completes Step 2. $\square$

To restate once more what we have proved:

II.F.15. PROPOSITION. *The winding number (II.F.6) and bounding-chain²³ (II.F.7)(v) definitions of null-homologous paths coincide.*

²²Each rectangle can be subdivided into a pair of simplices.

²³For $\gamma = \partial\mathcal{K}$, think of $\mathcal{K}$ as a finitely triangulable compact subset of U .

§ Homology Cauchy theorem.

Let γ be a closed 1-chain in a region U .

II.F.16. THEOREM. *If $f \in \mathcal{H}ol(U)$ and $\gamma \equiv 0$ in U , then*

$$\int_{\gamma} f(z) dz = 0.$$

PROOF. By (II.F.8), we may write $\gamma = \partial\mathcal{K}$, where

$$\mathcal{K} = \sum m_i \tau_i$$

is a 2-chain in U . Furthermore, we can make the τ_i as small as required (by triangulating) to fit in disks inside U :

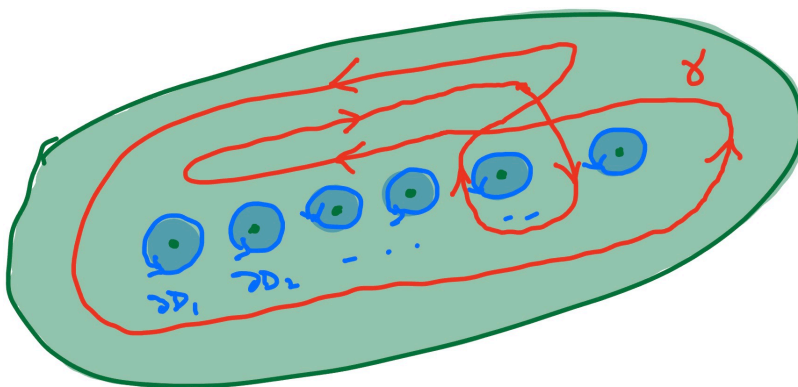
$$\tau_i \subset D_i \subset U.$$

Since each such $\partial\tau_i$ is homotopic to a point in D_i , hence in U , we have

$$\int_{\gamma} f dz = \int_{\sum m_i \partial\tau_i} f dz = \sum m_i \int_{\partial\tau_i} f dz = 0$$

by homotopy Cauchy II.C.10. □

Now let $z_1, \dots, z_m$ be points in U ; write $U^* = U \setminus \{z_1, \dots, z_m\}$ and let $D_1, \dots, D_m$ be “small” disks around the $\{z_i\}$.



Let $\gamma \subset U^*$ be a closed 1-chain, and assume that $\gamma \equiv 0$ in U (but not in U^*).

II.F.17. LEMMA. $\Gamma := \sum_i W(\gamma, z_i) \partial D_i - \gamma$ is homologous to 0 in U^* .

PROOF. Let $\alpha \in (U^*)^c$.

Case 1 $\alpha = z_j$ for some j : As $\gamma \mapsto \int_\gamma$ is linear, so is $\gamma \mapsto W(\gamma, z_0)$. (After all, the winding number is given by an integral.) Hence we may write

$$\begin{aligned} W(\Gamma, z_j) &= W\left(\sum_i W(\gamma, z_i) \partial D_i - \gamma, z_j\right) \\ &= \sum_i W(\gamma, z_i) \underbrace{W(\partial D_i, z_j)}_{\delta_{ij}} - W(\gamma, z_j) \\ &= W(\gamma, z_j) - W(\gamma, z_j) = 0. \end{aligned}$$

Case 2 $\alpha \in U^c$: Since $\gamma \equiv_{\text{hom}} 0$ in U , by definition $W(\gamma, \alpha) = 0$ for any α outside U . As boundaries of disks contained in U , the ∂D_i are $\equiv_{\text{hom}} 0$ too, whence $W(\partial D_i, \alpha) = 0$. Thus $W(\Gamma, \alpha) = 0$. $\square$

So in U^* , we have

$$(II.F.18) \quad \gamma \equiv_{\text{hom}} \sum_i W(\gamma, z_i) \partial D_i.$$

by rewriting the result of the Lemma. Thus, given a function $f \in \mathcal{H}ol(U^*)$, we have

$$(II.F.19) \quad \int_\gamma f(z) dz = \sum_i W(\gamma, z_i) \int_{\partial D_i} f(z) dz$$

by applying homology Cauchy II.F.16 to f , Γ , and U^* .

§ Cauchy integral formula (bis).

The “homology technology” leads to a much more convenient version of this important tool.

II.F.20. THEOREM (Cauchy Integral Formula, v. 3.0). *Let γ be a closed C^0 path in a region U , with $\gamma \equiv 0$ in U , and $f \in \mathcal{H}ol(U)$. Then*

$$\int_{\gamma} \frac{f(z)}{z - z_0} dz = 2\pi i W(\gamma, z_0) \cdot f(z_0).$$

PROOF. Since f is holomorphic on U , it is analytic at z_0 . Therefore on some small disk about z_0 , containing a smaller disk D around whose boundary we will integrate,

$$f(z) = \sum_{k \geq 0} a_k (z - z_0)^k.$$

We *cannot* substitute this in to $\int_{\gamma} \frac{f(z)}{z - z_0} dz$, as γ may lie outside the disk of convergence of the power series. But we *can* apply (II.F.19) to

$$\frac{f(z)}{z - z_0} \in \mathcal{H}ol(U \setminus \{z_0\})$$

to compute

$$\begin{aligned} \int_{\gamma} \frac{f(z)}{z - z_0} dz &= W(\gamma, z_0) \int_{\partial D} \frac{f(z)}{z - z_0} dz \\ &= W(\gamma, z_0) \sum_{k \geq 0} a_k \underbrace{\int_{\partial D} (z - z_0)^{k-1} dz}_{=0 \text{ for } k \geq 1} \\ &= W(\gamma, z_0) a_0 \int_{\partial D} \frac{dz}{z - z_0} \\ &= 2\pi i W(\gamma, z_0) a_0, \end{aligned}$$

which completes the proof since $a_0 = f(z_0)$. □

A more general result is:

For $f \in \mathcal{H}ol(U)$ and γ a C^0 closed path in $U \setminus \{z_1, \dots, z_m\}$ with $\gamma \equiv 0$,
hom

$$\text{we have } \int_{\gamma} \frac{f(z) dz}{(z - z_1) \cdots (z - z_m)} = 2\pi i \sum_{j=1}^m W(\gamma, z_j) f(z_j).$$

Right?

NO!!

What's wrong?

Turns out, $f(z_j)$ on the right hand side has to be replaced by

$$\frac{f'(z_j)}{(z_j - z_1) \cdots \widehat{(z_j - z_j)} \cdots (z_j - z_m)}.$$

We'll address this later (in the context of the residue theorem) in greater generality, though you could certainly write out a proof of it now.

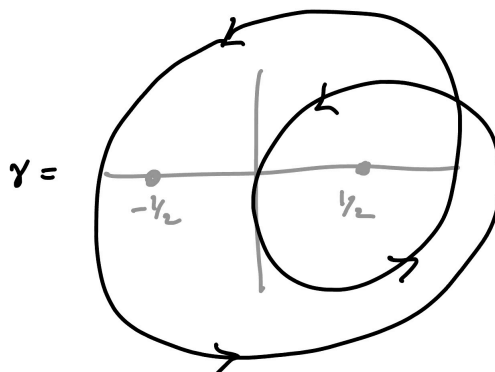
The exercises contain a simpler instance of multiple poles, the general case of which is (for $\gamma \equiv \text{in } U$)

$$(II.F.21) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \sum_i W(\gamma, z_i) \text{ord}_{z_i}(f),$$

where z_i runs through the zeroes of f in U . The heuristic idea behind this is that $\int_{\gamma} "d\log(f(z))"$ counts the number of times $i \arg(f(z))$ picks up $2\pi i$ as z travels along γ (if we don't make a branch cut), or equivalently the number of times $f(z)$ encircles 0. (The contribution made by each zero of f is computed by the product of the winding number with the multiplicity of the zero.) Hence, (II.F.21) is called the **argument principle**, which we'll also return to later.

We conclude with a simple application of II.F.20.

II.F.22. EXAMPLE. Consider the closed path shown:



In particular, we have $W(\gamma, \frac{1}{2}) = 2$ and $W(\gamma, -\frac{1}{2}) = 1$. Let

$$f(z) := \frac{e^{\pi iz}}{z^2 - \frac{1}{4}}.$$

Then we compute

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} \frac{e^{\pi iz} dz}{(z - \frac{1}{2})(z + \frac{1}{2})} \\ &= \int_{\gamma} \frac{e^{\pi iz}}{z - \frac{1}{2}} dz - \int_{\gamma} \frac{e^{\pi iz}}{z + \frac{1}{2}} dz \\ &= 2\pi i \left[W(\gamma, \frac{1}{2}) e^{\pi i(\frac{1}{2})} - W(\gamma, -\frac{1}{2}) e^{\pi i(-\frac{1}{2})} \right] \\ &= 2\pi i [2i - 1(-i)] = 2\pi i(3i) = -6\pi. \end{aligned}$$

Exercises

- (1) Let $f(z) = (z - z_0)^m h(z)$, where h is analytic on an open set U , and $h(z) \neq 0$ for all $z \in U$. Let γ be a closed path homologous to 0 in U , and such that z_0 does not lie on γ . Prove that $\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = W(\gamma, z_0)m$.
- (2) Let U be a simply connected region and let $z_1, \dots, z_n$ be points of U . Let $U^* = U \setminus \{z_1, \dots, z_n\}$ be the set obtained from U by deleting the points $z_1, \dots, z_n$, and f be analytic on U^* . Let γ_k be a small circle centered at z_k and let $a_k = \frac{1}{2\pi i} \int_{\gamma_k} f(\zeta) d\zeta$. Put $h(z) := f(z) - \sum \frac{a_k}{z - z_k}$. Prove that there exists an analytic function H on U^* such that $H' = h$.