

III. Applications of Cauchy's Theorems

III.A. Properties of analytic functions

§ A survey of local mapping theorems.

Let $U, V \subset \mathbb{C}$ be open sets, and $f: U \rightarrow V$ an analytic function. The theorems in this section will be proved in later sections.

III.A.1. DEFINITION. f is (globally analytically) **invertible** if there exists an analytic function $g: V \rightarrow U$ with $g \circ f = \text{Id}_U$ and $f \circ g = \text{Id}_V$.

Note that we are talking about *composition inverse*, not multiplicative inverse.

III.A.2. THEOREM. f is invertible $\iff f$ is injective and surjective.

The forward implication is clear since $g \circ f = \text{Id}_U \implies f$ is 1-to-1 and $f \circ g = \text{Id}_V \implies f$ is onto.

III.A.3. DEFINITION. f is (analytically) **locally invertible** at a point $z_0 \in U$ if there is an open subset $U_0 \subseteq U$ containing z_0 such that $f|_{U_0}: U_0 \rightarrow f(U_0)$ is invertible.

III.A.4. INVERSE MAPPING THEOREM (IMT). f is locally invertible at $z_0 \iff f'(z_0) \neq 0$.

We remark that $f'(z_0) \neq 0$ can also be written in the form

$$(III.A.5) \quad f(z) \underset{\text{loc}}{=} a_0 + a_1(z - z_0)(1 + h(z - z_0))$$

near z_0 , where $a_1 \neq 0$ and h is a power series (in $(z - z_0)$).

True or False: f locally invertible at every $z_0 \in U \iff f$ invertible.

False.

The implication " $\implies$ " doesn't hold: for instance,

- $f(z) = z^m, f: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C} \setminus \{0\}$;
- $f(z) = \exp(z), f: \mathbb{C} \rightarrow \mathbb{C} \setminus \{0\}$.

Recall that $U \subset \mathbb{C}$ is (topologically) connected if we *cannot* write $U = V \sqcup W$ with both V and W open (hence also closed in U).

For the remainder of the section let $f: U \rightarrow \mathbb{C}$ be an analytic function on a connected open set.

III.A.6. DEFINITION. f is **open** if it takes open sets (in U) to open sets (in $\mathbb{C}$).

III.A.7. OPEN MAPPING THEOREM (OMT). f is open $\iff f$ is nonconstant on U .

The right-hand condition is equivalent (for an analytic function) to having *some* $f^{(m)}(z_0) \neq 0$ (with $m \geq 1$) for each $z_0 \in U$. (Otherwise, f is represented by a constant power series about some point and is therefore constant on U .) Equivalently, for each $z_0 \in U$ we may represent f locally by

$$(III.A.8) \quad f(z) \underset{\text{loc}}{=} a_0 + a_m(z - z_0)^m(1 + h(z - z_0))$$

with $a_m \neq 0$. (Compare with (III.A.5).)

III.A.9. REMARK. If $w = f(z)$ in (III.A.8), then taking local analytic coordinates $\tilde{z} = (z - z_0)(1 + h(z - z_0))^{1/m}$ and $\tilde{w} = w - a_0$, then the mapping takes the form $\tilde{w} = \tilde{z}^m$. (Note that $(1 + h(z - z_0))^{1/m}$ is locally a well-defined analytic function after choosing a branch.)

§ The maximum modulus principle.

Here we continue to assume that f is analytic on a connected open set U . (If U is not connected, there is an obvious counterexample to the next result.)

III.A.10. MAXIMUM MODULUS PRINCIPLE (MMP). *Suppose that for some $z_0 \in U$, we have*

- (i) $|f(z_0)| \geq |f(z)|$,
- (ii) $\operatorname{Re}(f(z_0)) \geq \operatorname{Re}(f(z))$, or
- (iii) $\operatorname{Im}(f(z_0)) \geq \operatorname{Im}(f(z))$

for all $z \in U$. Then f is constant.

PROOF 1. Assume f is not constant; then by the OMT f is open. In particular, for $D = D(z_0, \delta) \subset U$, $f(D) \subseteq \mathbb{C}$ is open, hence contains $D(f(z_0), \epsilon)$ for some $\epsilon > 0$. Consider

$$w := f(z_0) + \begin{cases} \frac{\epsilon}{2} \exp(i \arg(f(z_0))) \\ \epsilon/2 \\ i\epsilon/2 \end{cases} \in f(D),$$

which has (respectively) greater modulus, Re, or Im than $f(z_0)$, yet (being in $f(D)$) is equal to $f(z_1)$ for some $z_1 \in D \subset U$. ✖ $\square$

PROOF 2 ((i) ONLY). For any sufficiently small $\epsilon > 0$,

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + \epsilon e^{i\theta})}{z_0 + \epsilon e^{i\theta} - z_0} i\epsilon e^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + \epsilon e^{i\theta}) d\theta \end{aligned}$$

whence

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + \epsilon e^{i\theta})| d\theta.$$

But since $|f(z_0 + \epsilon e^{i\theta})| \leq |f(z_0)|$, this forces equality

$$|f(z_0 + \epsilon e^{i\theta})| = |f(z_0)| \quad (\forall \theta, \epsilon).$$

Therefore f is constant on D , hence on U . $\square$

We turn to some immediate consequences.

III.A.11. COROLLARY. *Given $K \subseteq \mathbb{C}$ compact and $f: K \rightarrow \mathbb{C}$ continuous, with $f|_{\mathring{K}}$ analytic. Then $|f|$ attains its maximum over K on ∂K .*

PROOF. The composition $K \xrightarrow{f} \mathbb{C} \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0}$ exhibits $|f|$ as a continuous function on a compact set, which thereby *attains* its maximum $\sup_K |f|$ at some point $z_0 \in K$. Suppose $z_0 \in \mathring{K}$, since otherwise we are done. Let U denote the connected component of $\mathring{K}$ containing z_0 . Then K contains the closure $\overline{U}$, and boundary points of $\overline{U}$ are boundary points of K . The MMP says that f is constant on U , whence (by continuity) $|f|$ is constant and equal to $|f(z_0)|$ on $\overline{U}$. Hence this value is attained on ∂K . $\square$

We also get a new proof of the Fundamental Theorem of Algebra:

III.A.12. COROLLARY. *Any nonconstant polynomial has a root in $\mathbb{C}$.*

PROOF. Suppose $P(z)$ has no such root; then $\frac{1}{P}$ is entire. Since $\lim_{|z| \rightarrow \infty} \frac{1}{P(z)} = 0$, there exists an $R > 0$ such that $|z| \geq R \implies \left| \frac{1}{P(z)} \right| < \left| \frac{1}{P(0)} \right|$. But by Cor. III.A.11, $\frac{1}{P}$ must attain its maximum on $\overline{D}_R$ in ∂D_R . $\times$ $\square$

III.A.13. REMARK. An immediate consequence is that $P(z)$ (say, of degree n) factors as $C \cdot \prod_{i=1}^n (z - z_i)$. Indeed, by III.A.12 $P(z_0) = 0$ for some z_0 . By the division algorithm,

$$(III.A.14) \quad P(z) = (z - z_0)P_1(z) + r(z)$$

with $\deg r < \deg(z - z_0) = 1$. That is, $r(z) \equiv C_1$ is a constant, and substituting z_0 into (III.A.14) yields $0 = P(z_0) = r(z_0) = C_1$. Now keep going, with P_1 replacing P .

III.A.15. COROLLARY. *Let $\{z_i\}_{i=1}^n$ be points on the unit circle S^1 . Then the product of the distances $d(z, z_i)$ is at least 1 for some $z \in S^1$.*

PROOF. Consider $f(z) = \prod_{i=1}^n (z - z_i)$ as a function on the closed unit disk $\overline{\Delta}$. Then $|f(0)| = 1$, and f is obviously not constant. By Cor. III.A.11, $\max_{z \in \overline{\Delta}} |f(z)|$ is attained on $\partial \Delta$. $\square$

§ Proofs of the mapping theorems.

We will give two proofs of the local mapping theorems, one via the Cauchy integral formula (here) and another via power series methods (at the end of the chapter).

Let U be a region, $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$ nonconstant with $f(z_0) =: w_0$, and set

$$n := \text{ord}_{z_0}(f(z) - w_0).$$

Writing $D := D(z_0, \epsilon)$ and $\overline{D}^* = \overline{D} \setminus \{z_0\}$, we may choose $\epsilon > 0$ such that

$$\begin{cases} f'(z) \neq 0 \\ f(z) \neq w_0 \end{cases} \quad \text{on } \overline{D}^*,$$

since zeroes of $f'(z)$ and $f(z) - w_0$ can't have z_0 as an accumulation point. Set $\gamma := \partial D$.

III.A.16. THEOREM. *There exists $\delta > 0$ such that*

$$0 < |\alpha - w_0| < \delta \implies f(z) = \alpha \text{ has exactly } n \text{ solutions in } D.$$

PROOF. Denote the solutions $\{z_j(\alpha)\}_{j=1}^{\mu(\alpha)} \subset D^*$. Each has multiplicity one, since

$$(f(z) - \alpha)' = f'(z) \neq 0 \quad \text{at } z_j(\alpha).$$

So

$$f(z) - \alpha = \left(\prod_{j=1}^{\mu(\alpha)} (z - z_j(\alpha)) \right) g(z)$$

with g holomorphic and nonvanishing on D , and

$$\begin{aligned} \mu(\alpha) &= \sum_{j=1}^{\mu(\alpha)} 1 = \sum_{j=1}^{\mu(\alpha)} W(\gamma, z_j(\alpha)) \\ &= \sum_{j=1}^{\mu(\alpha)} \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_j(\alpha)} + \frac{1}{2\pi i} \int_{\gamma} \frac{g'(z)}{g(z)} dz \quad \xrightarrow{\text{0 by Cauchy}} \\ &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z) dz}{f(z) - \alpha} \\ &= \frac{1}{2\pi i} \int_{f(\gamma)} \frac{dw}{w - \alpha} = W(f(\gamma), \alpha), \end{aligned}$$

which is constant on components of $\mathbb{C} \setminus f(\gamma)$. (Note that the latter is an open set, since $f(\gamma)$ is closed.) By assumption (that $f(z) \neq w_0$ on $\overline{D}^*$), we have $w_0 \notin f(\gamma)$.

So let V denote the component of $\mathbb{C} \setminus f(\gamma)$ containing w_0 , and take $\delta > 0$ small enough that $D(w_0, \delta) \subset V$. Now two things are true:

- the winding number $W(f(\gamma), w)$ is constant for $w \in V$; and
- $W(f(\gamma), w_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)dz}{f(z) - w_0} = \text{ord}_{z_0}(f(z) - w_0) = n$.¹

Thus $W(f(\gamma), w) \equiv n$ for $w \in V$, and in particular $\mu(\alpha) = n$ for $\alpha \in D^*(w_0, \delta)$. $\square$

The three mapping theorems are now corollaries of III.A.16:

PROOF OF III.A.7 (OMT). We need to establish that f as above sends open sets to open sets. It suffices to show that for arbitrarily small discs D about arbitrary $z_0 \in U$, $f(D)$ contains a small open disk. (Why?) But $f(D) \supset V \supset D(w_0, \delta)$ in the above proof. $\square$

PROOF OF III.A.4 (IMT). We must show that if $f'(z_0) \neq 0$, then f is a local analytic isomorphism. The hypothesis forces $n = 1$, and Theorem III.A.16 implies that f has a single-valued inverse in $D(w_0, \delta)$. This inverse is continuous because f is open. It is then analytic simply by, at each point, taking the reciprocal of the limit computing f' (yielding of course “ $1/(f' \circ f^{-1})$ ”). $\square$

PROOF OF III.A.2. If $f: U \rightarrow V$ is analytic and bijective, then there is a set-theoretic inverse $g: V \rightarrow U$ which is $\mathcal{C}^0$ by the OMT. To show g is analytic, apply Theorem III.A.16 at any point z_0 : bijectivity $\implies n = 1 \implies f'(z_0) \neq 0$. Now apply IMT, done. $\square$

¹See the first exercise of Chapter II.F.

§ The formal composition inverse.

Suppose we are given a formal power series like

$$f(T) = T + 12T^2 - 60T^3 + \dots$$

and we seek $g(T) = T + b_2T^2 + b_3T^3 + \dots$ such that

$$\begin{aligned} T &= (f \circ g)(T) := g(T) + 12(g(T))^2 - 60(g(T))^3 + \dots \\ &= T + (b_2 + 12)T^2 + (b_3 + 24b_2 - 60)T^3 + \dots \end{aligned}$$

Clearly we must take $b_2 = -12$ and $b_3 = 348$. More generally:

III.A.17. THEOREM. *Suppose $f(T)$ is a formal power series with zero constant term and nonzero coefficient of T (so that the corresponding function has nonzero derivative at the origin). Then a two-sided formal power series composition inverse $g(T)$ exists; that is, $f(g(T)) = T = g(f(T))$.*

PROOF. After rescaling f becomes

$$(III.A.18) \quad f(T) = T - \sum_{n \geq 2} a_n T^n,$$

and we aim to choose $\{b_k\}_{k \geq 1}$ so that

$$(III.A.19) \quad T = (f \circ g)(T) = \sum_{k \geq 1} b_k T^k - \sum_{n \geq 2} a_n \left(\sum_{k \geq 1} b_k T^k \right)^n$$

holds. Now taking I to range over multi-indices $I = \{i_1, \dots, i_{|I|}\}$ and $\mathfrak{F}$ to range over functions $\mathfrak{F}: I \rightarrow \mathbb{N}$ with $\sum_{i \in I} \mathfrak{F}(i) = n$,

$$\begin{aligned} \sum_{n \geq 2} a_n \left(\sum_{k \geq 1} b_k T^k \right)^n &= \sum_{n \geq 2} a_n \left\{ \sum_{I, \mathfrak{F}} \left(\prod_{i \in I} b_i^{\mathfrak{F}(i)} \right) \frac{n!}{\mathfrak{F}(i_1)! \cdots \mathfrak{F}(i_{|I|})!} T^{\sum_{i \in I} i \mathfrak{F}(i)} \right\} \\ &= \sum_{n \geq 2} a_n \sum_{k \geq 2} \sum'_{I, \mathfrak{F}} \left(\prod_I b_i^{\mathfrak{F}(i)} \right) \frac{n!}{\mathfrak{F}(i_1)! \cdots \mathfrak{F}(i_{|I|})!} T^k \end{aligned}$$

[where the $\sum'$ is subject to $\sum_I \mathfrak{F}(i) = n$ and $\sum_I i \mathfrak{F}(i) = k$]

$$(III.A.20) \quad = \sum_{k \geq 2} T^k \left\{ \sum''_{I, \mathfrak{F}} a_{\sum_I \mathfrak{F}(i)} \left(\prod_I b_i^{\mathfrak{F}(i)} \right) \frac{(\sum \mathfrak{F}(i))!}{\mathfrak{F}(i_1)! \cdots \mathfrak{F}(i_{|I|})!} \right\}$$

[where the (finite!) $\sum''$ is subject to $\sum_I \mathfrak{F}(i) \geq 2$ and $\sum_I i \mathfrak{F}(i) = k$].

Here we remark that the condition $\sum_I \mathfrak{F}(i) \geq 2$ ensures that every $i \in I$ is strictly less than k . It is this that makes it possible to define b_k inductively to be the quantity in braces in (III.A.20), which makes (III.A.19) hold. This establishes that a (right) composition inverse g of f always exists.

Writing $h(T)$ for the formal (right) composition inverse of g , so that $g(h(T)) = T$, we have

$$g(f(T)) = g(f(g(h(T)))) = g(h(T)) = T.$$

Thus g is also a left composition inverse to f (and $f = h$). □

In the next section, this leads to a new proof of the IFT and OMT.

Here is a sample use of the formula for the coefficients $\{b_k\}$ of the inverse power series of $f(z) = \sum_{n \geq 1} a_n z^n$ with $a_1 = 1$:

- $b_1 = 1$; and
- summing over $I, \mathfrak{F}$ with $\sum_{i \in I} \mathfrak{F}(i) \geq 2$ and $\sum_{i \in I} i \mathfrak{F}(i) = k$,²

$$-b_k = \sum_{I, \mathfrak{F}}'' a_{\sum \mathfrak{F}(i)} \left(\prod_I b_i^{\mathfrak{F}(i)} \right) \frac{(\sum \mathfrak{F}(i))!}{\mathfrak{F}(i_1)! \cdots \mathfrak{F}(i_{|I|})!}$$

(Think of $\sum_{i \in I} i \mathfrak{F}(i)$ as a partition of k .) This is a little ugly, but sometimes more efficient than the alternative (writing out $z = (f \circ g)(z)$).

III.A.21. EXAMPLE. Consider $f(z) = \sin(z)$. Then

$$a_1 = 1, a_2 = 0, a_3 = -\frac{1}{6}, a_4 = 0, a_5 = \frac{1}{120}, \dots$$

Obviously $b_{\text{even}} = 0$ (so that we only need to consider partitions into *odd numbers*), and $b_1 = 1$. The first interesting b_i is

$$b_3 = -a_3 b_1^3 \frac{3!}{3!} = -\left(-\frac{1}{6}\right) 1^3 \cdot 1 = \frac{1}{6} = \frac{1}{2 \cdot 3}$$

where the only relevant partition is $3 = 1 + 1 + 1$. The next is

$$b_5 = -a_5 b_1^5 \frac{5!}{5!} - a_3 b_1^2 b_3 \frac{3!}{2!1!} = -\frac{1}{120} + \frac{1}{6} \cdot \frac{3}{6} = \frac{3}{40} = \frac{1 \cdot 3}{2 \cdot 4 \cdot 5},$$

where we used the two partitions $1 + 1 + 1 + 1 + 1$ and $3 + 1 + 1$ of 5. (Of course, there is a better way to get the power series of arcsin.)

²Think of $\sum_I i \mathfrak{F}(i) = k$ as a partition of k ($\mathfrak{F}$ is "frequency").

§ Mapping theorems via power series.

Using the notation of the last section, if f has nonzero radius of convergence r_f , then $|a_n| \leq A^n$ ($\forall n$).

III.A.22. CLAIM. *The formal composition inverse g has radius of convergence $r_g > 0$.*

PROOF. Set $F(T) := T - \sum_{n \geq 2} A^n T^n$, with formal composition inverse $G(T) := \sum B_k T^k$. Inductively assuming $B_\ell \geq |b_\ell|$ ($\forall \ell < k$), we have

$$\begin{aligned} B_k &= \sum_{I, \mathfrak{F}}'' A^{\sum \mathfrak{F}(i)} \left(\prod_{i \in I} B_i^{\mathfrak{F}(i)} \right) \frac{(\sum \mathfrak{F}(i))!}{\mathfrak{F}(i_1)! \cdots \mathfrak{F}(i_{|I|})!} \\ &\geq \sum_{I, \mathfrak{F}}'' |a_{\sum \mathfrak{F}(i)}| \left(\prod_I |b_i|^{\mathfrak{F}(i)} \right) \frac{(\sum \mathfrak{F}(i))!}{\mathfrak{F}(i_1)! \cdots \mathfrak{F}(i_{|I|})!} \\ &\geq \left| \sum_{I, \mathfrak{F}}'' a_{\sum \mathfrak{F}(i)} \left(\prod_I b_i^{\mathfrak{F}(i)} \right) \frac{(\sum \mathfrak{F}(i))!}{\mathfrak{F}(i_1)! \cdots \mathfrak{F}(i_{|I|})!} \right| = |b_k| \end{aligned}$$

where the last equality is from the proof of Theorem III.A.17. So $\limsup |B_k|^{1/k} \geq \limsup |b_k|^{1/k} \implies r_g > r_G$, and it will suffice to show $r_G > 0$.

Now (working formally)

$$T = (F \circ G)(T) = G - \sum_{n \geq 2} (AG)^n = G - \frac{A^2 G^2}{1 - AG},$$

and multiplying by $(1 - AG)$ gives

$$0 = (A^2 + A)G^2 - (1 + AT)G + T.$$

Finally we “solve” the quadratic equation (understood formally, in the sense of *series* solving an equation) to obtain

$$G(T) = \frac{(1 + AT)}{2A(1 + A)} \left\{ 1 - \left(1 - \frac{4T(A^2 + A)}{(1 + AT)^2} \right)^{1/2} \right\}.$$

The $(\dots)^{1/2}$ can be understood as the composition of formal power series with *positive* radius of convergence, and so $r_G > 0$. $\square$

Now we apply this to the corresponding *analytic functions* $f(z)$ and $g(z)$, where by Prop. I.D.27 $(f \circ g)(T) = T = g(f(T)) \implies f(g(z)) = z = g(f(z))$ on some sufficiently small disks.

(RE-)PROOF OF III.A.4 AND III.A.7. More precisely, by continuity of $g(z)$, there exists $\epsilon > 0$ such that $V := D(0, \epsilon)$ has $g(V) \subset D(0, r_f)$. Put $U := f^{-1}(V)$, so that f maps $U \rightarrow V$. Given $v \in V$, $f(g(v)) = v \implies g(v) \in f^{-1}(V) = U$, so g maps $V \rightarrow U$ and we have f, g local analytic isomorphisms at 0. This yields the " $\Leftarrow$ " part of the IMT.

Now let $\text{ord}(f) = m \geq 1$. (Until now, it's been $m = 1$.) That is, writing $0 \neq a_m =: \alpha^m$,

$$(III.A.23) \quad f(z) = a_m z^m (1 + h(z)) = (\alpha(1 + h_1(z)))^m =: (f_1(z))^m$$

for some local holomorphic functions h and h_1 with $h(0) = 0 = h_1(0)$. The resulting $f_1(z) = \alpha z + \{\text{higher order terms}\}$ is a local analytic isomorphism at 0 by the above argument. By continuity of the inverse, f_1 is *open* near 0. Since $w \mapsto w^m$ is also open, the composition f is open. Any nonconstant f sending $0 \mapsto 0$ is of the form (III.A.23), and so we recover the OMT.

Finally, if $m > 1$, then f cannot (locally) be an injection, since $w \mapsto w^m$ isn't; hence f will not be locally invertible, proving the " $\implies$ " part of IMT. $\square$

Exercises

- (1) Determine which of the following functions give local analytic isomorphisms at the origin: (i) $f(z) = e^z$; (ii) $f(z) = (\sin z)^2$; and (iii) $f(z) = z^2 + z$. For those that do, determine the largest disk about the origin on which f remains injective.
- (2) Find the coefficients of degree ≤ 7 in the power series expansion of $\tan(z)$ about $z = 0$, by finding the composition inverse of the series for $\arctan$.