

III.B. Multiply connected regions

The aim of this short chapter is to apply winding numbers to reduce all closed contour integrals of closed differentials on a region U to the computation of a period matrix which describes a pairing between first homology and cohomology of U . For us this is more important as a conceptual result; in practice we will not compute these matrices.

§ Cycles and connectivities.

Let U be an open set in $\mathbb{C}$ which is the union of finitely many regions. Recall that, denoting $\mathcal{C}^1$ k -chains by $C_k(U)$, we have boundary homomorphisms

$$0 \xrightarrow{\partial} C_2(U) \xrightarrow{\partial} C_1(U) \xrightarrow{\partial} C_0(U) \xrightarrow{\partial} 0$$

satisfying $\partial \circ \partial = 0$, and the **singular homology groups** of U are

$$H_k(U, \mathbb{Z}) := \frac{\ker(\partial) \subset C_k(U)}{\operatorname{im}(\partial) \subset C_k(U)};$$

often we just write $H_k(U)$. A k -chain in $\ker(\partial)$ is **closed**, or a **k-cycle**. Let $h_i(U) := \operatorname{rank}(H_i(U))$.

III.B.1. PROPOSITION. *Let m be the number of connected components of U . Then $h_0(U) = m$.*

PROOF. Write $U = U_1 \sqcup \cdots \sqcup U_m$ for the decomposition into connected components, and define a map

$$\phi_0: H_0(U) \rightarrow \mathbb{Z}^m$$

by sending the class $[p]$ of a point $p \in U_i$ to the i^{th} standard basis vector e_i . This is well-defined since a path γ with $\partial\gamma = p - q$ only exists if p and q belong to the same U_i , and such path boundaries generate the equivalence relation $\equiv_{\text{hom}}$. For injectivity, if $\phi_0(\sum n_p[p]) = 0$, then $\sum_{p \in U_i} n_p = 0$ for each i and so there exists a 1-chain γ “connecting the dots” in each U_i , with $\partial\gamma = \sum n_p \cdot p$, whence $\sum n_p[p] \equiv_{\text{hom}} 0$. Since ϕ_0 is obviously surjective, it is an isomorphism and we are done. $\square$

In particular, we conclude that

$$U \text{ is connected} \iff h_0(U) = 1.$$

Assume now that U is connected, and consider $h_1(U)$.

III.B.2. DEFINITION. The **connectivity** of U is the number of connected components of $\hat{\mathbb{C}} \setminus U (= U^c \cup \{\infty\})$.

III.B.3. PROPOSITION. Let n denote the connectivity of U . Then we have $h_1(U) = n - 1$.

PROOF. Write the complement $\hat{\mathbb{C}} \setminus U = V_1 \sqcup \cdots \sqcup V_{n-1} \sqcup \hat{V}_n$, where $\hat{V}_n$ is the unbounded component³ and we put $V_n := \hat{V}_n \setminus \{\infty\}$. For each $i < n$, let $\alpha_i \in V_i$ be any point, and define

$$\begin{aligned} \phi_1: H_1(U) &\rightarrow \mathbb{Z}^{n-1} \\ [\gamma] &\mapsto (W(\gamma, \alpha_1), \dots, W(\gamma, \alpha_{n-1})) \end{aligned}$$

This is *well-defined* because $\gamma = \partial K \implies W(\gamma, \alpha) = 0 \ \forall \alpha \in U^c$ by (II.F.8). For injectivity, suppose $W(\gamma, \alpha_i) = 0 \ (\forall i < n)$. Then $W(\gamma, \alpha) = 0$ for any α in any V_i , since

- $W(\gamma, z) = 0$ for z in the unbounded component (cf. II.E.20), and
- $W(\gamma, z)$ is constant on connected components of $\mathbb{C} \setminus \gamma$ (and each V_i is contained in one such component).

Now use $W(\gamma, \alpha) = 0 \ (\forall \alpha \in U^c) \implies \gamma = \partial K \implies [\gamma] = 0$, once again using (II.F.8).

For the surjectivity, cover $\mathbb{C}$ with squares sufficiently small that no square meets more than one V_i . (Let $\mu = \min_{i \neq j} d(V_i, V_j)$, which is > 0 as the $V_{i < n}$ are compact, and use squares of side-length $\frac{1}{2}\mu$.) Let $\mathfrak{W}_i$ be the union of (closed) squares meeting V_i ; then $V_i \subset \mathfrak{W}_i$ while $\partial \mathfrak{W}_i \cap V_i = \emptyset = \mathfrak{W}_i \cap V_{j \neq i}$. Consequently $\partial \mathfrak{W}_i \subset U$ and, by considering integrals over the squares in $\mathfrak{W}_i$,

$$W(\partial \mathfrak{W}_i, a_j) = \delta_{ij} \quad (i, j < n).$$

So the image of ϕ_1 hits all the e_i vectors and we're done. $\square$

³Recall that this is the component containing ∞ . It is possible that this component consists only of ∞ , in which case V_n is empty.

We may conclude from this Remark and the surjectivity of the Hurewicz map (cf. II.F.10) that

(III.B.5)

Taking $\gamma_i := \partial \mathfrak{W}_i$ from the proof of Prop. III.B.3, we see that

One can also show that $\pi_1(U)$ is the free group on these γ_i .⁴ Hence, for any closed path γ in U , we have

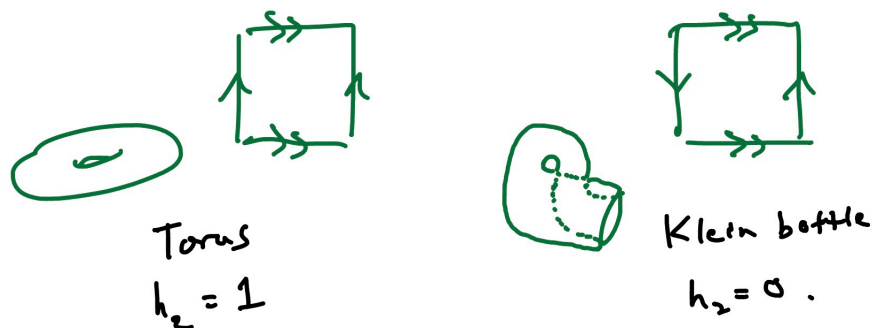
$$(III.B.6) \quad \gamma \equiv \sum_{i=1}^{n-1} a_i \gamma_i \text{ (in } U), \text{ for some } a_i \in \mathbb{Z}.$$

To get $H_2(U)$ nonzero, you first need to replace U by a *compact* real 2-manifold M (e.g., a Riemann surface). Still assuming that this

⁴i.e. elements of π_1 are the *words* in $n - 1$ generators. In contrast, for a compact Riemann surface of genus g , $H_1(M) = \mathbb{Z}\langle \gamma_1, \dots, \gamma_{2g} \rangle$, but $\pi_1(M)$ is the *quotient* of the free group on $\gamma_1, \dots, \gamma_{2g}$ by the relation $[\gamma_1, \gamma_{g+1}] \cdot [\gamma_2, \gamma_{g+2}] \cdots [\gamma_g, \gamma_{2g}] = 1$.

M is connected, it is said to be **orientable** if $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$. In this case H_2 is spanned by the fundamental class $[M]$, i.e. the class of a finite triangulation of M .

III.B.7. EXAMPLE. Consider the manifolds obtained by identifying sides of the square as shown:



The non-orientability of the Klein bottle can be checked by triangulating it and checking that the boundaries of the triangles don't cancel.

§ Periods of closed differentials.

So we have now had a more thorough look at chains and homology. Next we have to examine differential forms and (de Rham) cohomology. Denoting

$$A^k(U) := \mathcal{C}^\infty k\text{-forms on } U,$$

the exterior derivative (cf. Chapter II.B) gives homomorphisms

$$0 \xrightarrow{d} A^0(U) \xrightarrow{d} A^1(U) \xrightarrow{d} A^2(U) \xrightarrow{d} 0,$$

We remind the reader that

- A^0 denotes $\mathcal{C}^\infty$ functions,
- A^1 consists of expressions $f dx + g dy$ (or $F dz + G d\bar{z}$), and
- A^2 consists of expressions $H dx \wedge dy$ (or $H dz \wedge d\bar{z}$),

where $f, g, F, G, H \in \mathcal{C}^\infty(U)$.

Note in particular that $d \circ d = 0$:

$$\begin{aligned} dF &= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy, \quad d(f dx + g dy) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy \\ \implies d(dF) &= \left(\frac{\partial^2 F}{\partial x \partial y} - \frac{\partial^2 F}{\partial y \partial x} \right) dx \wedge dy = 0. \end{aligned}$$

Thus it makes sense to define the **de Rham cohomology groups** of U by

$$H_{\text{dR}}^k(U) := \frac{\ker(d) \subset A^k(U)}{\text{im}(d) \subset A^k(U)} = \frac{\text{closed forms}}{\text{exact forms}}$$

Clearly

- $H_{\text{dR}}^0(U) = \{\text{constant functions}\} \cong \mathbb{C}$,

and it is also true (but we won't prove) that

- $H_{\text{dR}}^2(U) = \{0\}$ because all the 2-forms are exact.

Now $\ker(d) \subset A^1(U)$ consists (by definition) of the **closed 1-forms**:

$$\omega = A(x, y)dx + B(x, y)dy = f(z)dz + g(z)d\bar{z}$$

(with $A, B, f, g \in C^\infty(U)$) such that

$$0 = d\omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy = \left(\frac{\partial g}{\partial z} - \frac{\partial f}{\partial \bar{z}} \right) dz \wedge d\bar{z}$$

(i.e., $\frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$ or equivalently $\frac{\partial g}{\partial z} = \frac{\partial f}{\partial \bar{z}}$). In particular,

$$f(z) dz \text{ is closed } \iff f \in \mathcal{H}\mathcal{O}\mathcal{L}(U);$$

these are the **holomorphic 1-forms**.

III.B.8. POINCARÉ LEMMA. *Closed 1-forms are the same as locally exact⁵ 1-forms, i.e. those with a (not necessarily holomorphic) primitive in a small disk about each point.*

PROOF. Given ω locally exact, we may write in a neighborhood of any point in U

$$\omega \underset{\text{loc}}{=} dF = \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial \bar{z}} d\bar{z} =: f dz + g d\bar{z}$$

⁵Warning: this is not the same as global exactness, i.e. membership in $d(A^0(U))$.

for some $\mathcal{C}^\infty$ function F on the neighborhood. But then $\frac{\partial f}{\partial \bar{z}} = \frac{\partial g}{\partial \bar{z}}$, and ω is closed in the neighborhood. Since the point was arbitrary we have $d\omega = 0$ everywhere.

Conversely, given ω closed,

$$F(z) := \int_{z_0}^z \omega$$

is well-defined on a small disk about $z_0 \in U$, because (for $\gamma = \partial\Gamma$, $\Gamma \subset U$)

$$\int_{\gamma} A dx + B dy = \int_{\Gamma} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy = 0$$

by closedness. You can check by FTC that $dF = A dx + B dy = \omega$. $\square$

III.B.9. THEOREM. *The pairing*

$$\begin{aligned} H_1(U) \times H_{dR}^1(U) &\longrightarrow \mathbb{C} \\ ([\gamma], [\omega]) &\longmapsto \int_{\gamma} \omega \end{aligned}$$

is well-defined. In other words, $\int_{\gamma} \omega = 0$ if (i) $\gamma = \partial\mathcal{K}$ (i.e., $\gamma \equiv_{\text{hom}} 0$) or (ii) $\omega = dG$ (ω globally exact).

PROOF. (i) Triangulate $\mathcal{K}$, so that $\gamma = \sum n_i \partial\tau_i$; use local exactness of ω to see $\int_{\partial\tau_i} \omega = 0$ (as in the proof of Poincaré).

(ii) Multivariate fundamental theorem of calculus:

$$\int_{\gamma} dG = G(\gamma(b)) - G(\gamma(a)) = 0$$

since $\gamma(b) = \gamma(a)$ (γ closed). $\square$

Letting $\alpha_j \in V_j$ and γ_i be as in the proof of Prop. III.B.3, consider the forms

$$\omega_j = \frac{1}{2\pi i} \frac{dz}{z - \alpha_j}.$$

Writing $H_1(U, \mathbb{C}) := H_1(U, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, since

$$\int_{\gamma_i} \omega_j = W(\gamma_i, \alpha_j) = \delta_{ij},$$

we see that the pairing

$$H_1(U, \mathbb{C}) \times H_{\text{dR}}^1(U) \rightarrow \mathbb{C}$$

is in fact perfect, so that the **singular cohomology**

$$H^1(U, \mathbb{C}) := \text{Hom}(H_1(U, \mathbb{Z}), \mathbb{C})$$

is isomorphic to $H_{\text{dR}}^1(U)$. This is a special case of *de Rham's theorem*, which says more generally that for a smooth manifold M

$$H^k(M, \mathbb{C}) := \text{Hom}(H_k(M, \mathbb{Z}), \mathbb{C}) \cong H_{\text{dR}}^k(M).$$

The upshot of all this (for our purposes) is the extremely simple consequence of (i) in the last Theorem (applied to $\gamma - \sum n_i \gamma_i$):

III.B.10. COROLLARY. *Let U be a region with connectivity n , so that $H_1(U) \cong \mathbb{Z}\langle[\gamma_1], \dots, [\gamma_{n-1}]\rangle$ (free abelian group on $n - 1$ generators). Then for any 1-cycle γ in U and closed 1-form $\omega \in A^1(U)$ we have*

- $\gamma \equiv_{\text{hom}} \sum_{i=1}^{n-1} a_i \gamma_i$ for some integers $\{a_i\}_{i=1}^{n-1}$, and
- $\int_{\gamma} \omega = \sum_{i=1}^{n-1} a_i \int_{\gamma_i} \omega$.

This says that to know *any* closed-path integral of ω , you need only know its **periods** $\int_{\gamma_i} \omega$ over a homology basis. One can also think of this result as generalizing the homology Cauchy theorem from holomorphic 1-forms to all closed 1-forms.

Riemann initiated the systematic study of periods on algebraic curves, and the modern theory most closely associated with it is Hodge theory. There is also a fascinating article (available online) called “Periods” by M. Kontsevich and D. Zagier, which shows how they are really at the center of modern number theory as well.

Exercises

- (1) Prove that the region obtained from a simply connected region by removing m points has connectivity $m + 1$, and find a homology basis.
- (2) Show that a single-valued branch of $\sqrt{1 - z^2}$ can be defined in any region such that the points ± 1 are in the same component of

the complement. What are the possible values of $\int \frac{dz}{\sqrt{1-z^2}}$ over a closed curve in the region?