

III.C. The Schwarz awakens

§ The Schwarz lemma.

This is a key application of the Maximum Modulus Principle, whose proof was closely related to the Open Mapping Theorem.

Stepping beyond the primarily local considerations of Chapter III.A, the next challenge is to identify all the analytic isomorphisms of the unit disk $\Delta = D(0, 1)$, written

$$\Delta \xrightarrow[\cong]{\Xi} \Delta \quad \text{or} \quad \Xi \in \text{Aut}(\Delta).$$

Given a simply connected region U and an analytic isomorphism $f: U \xrightarrow{\cong} \Delta$, we'll then have all automorphisms of U : indeed, for each $\xi \in \text{Aut}(U)$, the composition

$$\Delta \xrightarrow{f^{-1}} U \xrightarrow{\xi} U \xrightarrow{f} \Delta$$

gives an automorphism of Δ , namely

$$\Xi = f \circ \xi \circ f^{-1}.$$

Likewise $\Xi \mapsto f^{-1} \circ \Xi \circ f$ goes the other way, and we get an isomorphism of groups

$$\text{Aut}(U) \cong \text{Aut}(\Delta).$$

The key result in the computation of $\text{Aut}(\Delta)$ is the following:

III.C.1. SCHWARZ LEMMA. *Suppose $f: \Delta \rightarrow \Delta$ is analytic,⁶ with $f(0) = 0$. Then:*

- (i) $|f(z)| \leq |z|$ ($\forall z \in \Delta$).
- (ii) $|f'(0)| \leq 1$.
- (iii) *If $|f(z_0)| = |z_0|$ for any $z_0 \in \Delta^*$, then $f(z) = e^{i\theta}z$ ($\theta \in \mathbb{R}$).*
- (iv) *If $|f'(0)| = 1$, then $f(z) = f'(0)z$.⁷*

PROOF. Every $f \in \mathcal{Hol}(\Delta)$ has a power series representation $f(z) = \sum_{n \geq 0} a_n z^n$, valid on (all of) Δ , with $\limsup_{n \rightarrow \infty} |a_n|^{1/n} \leq 1$.

⁶Note: f is *not* necessarily 1-to-1 or onto!

⁷In cases (iii) and (iv), the outcome is that f is a rotation.

Our hypothesis that $f(0) = 0$ makes $a_0 = 0$, and so

$$\begin{aligned} g(z) &:= \sum_{m \geq 0} a_{m+1} z^m \ (\in \mathcal{Hol}(\Delta)) \\ &= \begin{cases} f(z)/z, & z \in \Delta^* \\ f'(0), & z = 0 \end{cases} \end{aligned}$$

To prove (i), we must show $|g(z)| \leq 1$ ($\forall z \in \Delta$). Now

$$\begin{aligned} f(\Delta) \subseteq \Delta &\implies |f(z)| < 1 \ (\forall z \in \Delta) \\ &\implies |g(z)| < \frac{1}{|z|} \ (\forall z \in \Delta). \end{aligned}$$

Let $z_0 \in \Delta$, $r \in [|z_0|, 1)$ be arbitrary, and write $\overline{D}_r = \overline{D}(0, r)$. Then $g \in \mathcal{Hol}(\overline{D}_r)$, while $\|g(z)\|_{\partial D_r} < \frac{1}{r}$. Since $z_0 \in \overline{D}_r$, the MMP implies

$$|g(z_0)| \leq \|g(z)\|_{\partial D_r} < \frac{1}{r},$$

and taking $r \rightarrow 1^-$, we get $|g(z_0)| \leq 1$ as desired.

(ii) Taking $z_0 = 0$ now immediately gives $|f'(0)| = |g(0)| \leq 1$.

(iii) If $|f(z_0)| = |z_0|$ for some $z_0 \in \Delta^*$, then $|g(z_0)| = 1$ ($\implies g(z_0) = e^{i\theta}$ for some $\theta \in \mathbb{R}$). From (i), $\|g\|_{\Delta} \leq 1$, so the maximum modulus is achieved at $z_0 \in \Delta$ and g is constant $\equiv g(z_0) = e^{i\theta}$ by the MMP.

(iv) Finally, if $|g(0)| = 1$, then by (iii) $g \equiv g(0)$ hence $f(z)/z \equiv f'(0)$, i.e. $f(z) = f'(0)z$. $\square$

The statement and argument above are due to Caratheodory in 1907. Schwarz's original result in 1870 assumed that f was 1-to-1.

§ Automorphisms of the unit disk.

Now we want to compute the group of conformal automorphisms of Δ , i.e. $\text{Aut}(\Delta)$. The Schwarz Lemma describes those which fix the origin: they are the rotations. To see this, apply (ii) to f and f^{-1} to get $|f'(0)| = 1$, then apply (iv). But what about those not fixing 0?

Consider the fractional linear transformation

$$(III.C.2) \quad \phi_\alpha(z) := \frac{z - \alpha}{1 - \bar{\alpha}z}, \quad \alpha \in \Delta.$$

As a function on Δ , this is holomorphic, as the denominator is never zero. Moreover,

$$|z| = 1 \implies |\phi_\alpha(z)| = \left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right| = \left| \frac{z - \alpha}{\bar{z}(1 - \bar{\alpha}z)} \right| = \left| \frac{z - \alpha}{\bar{z} - \bar{\alpha}} \right| = 1;$$

and so the MMP says that $|\phi_\alpha(z)| \leq 1$ on Δ , with equality at some point iff ϕ_α is constant. Since ϕ_α is visibly *not* constant,

$$|z| < 1 \implies |\phi_\alpha(z)| < 1.$$

Now writing

$$\begin{aligned} w = \frac{z - \alpha}{1 - \bar{\alpha}z} &\implies w - \bar{\alpha}zw = z - \alpha \\ &\implies w + \alpha = z(1 + \bar{\alpha}w) \\ &\implies z = \frac{w + \alpha}{1 + \bar{\alpha}w} = \phi_{-\alpha}(w), \end{aligned}$$

we find that

$$(III.C.3) \quad \phi_\alpha^{-1} = \phi_{-\alpha}.$$

Conclude that

$$(III.C.4) \quad \phi_\alpha \in \text{Aut}(\Delta).$$

Another type of element of $\text{Aut}(\Delta)$ is $\mu_{e^{i\theta}}$, where $\theta \in \mathbb{R}$ and

$$\mu_{e^{i\theta}}(z) := e^{i\theta}z.$$

III.C.5. THEOREM. $F \in \text{Aut}(\Delta) \implies \exists \alpha \in \Delta, \theta \in \mathbb{R}$ such that

$$F = \phi_\alpha \circ \mu_{e^{i\theta}} (= \mu_{e^{i\theta}} \circ \phi_{\alpha e^{-i\theta}}).$$

PROOF. Use the Schwarz: let $\beta := F(0)$; applying Schwarz (ii) to $f := \phi_\beta \circ F$ (which sends $0 \mapsto 0$) yields $|f'(0)| \leq 1$. Applying it to $f^{-1} = F^{-1} \circ \phi_{-\beta}$, we get $|\frac{1}{f'(0)}| = |(f^{-1})'(0)| \leq 1$. So $|f'(0)| = 1$, which by Schwarz (iv) $\implies (\phi_\beta \circ F) = f = \mu_{e^{i\theta}}$ for some $\theta \in \mathbb{R} \implies F = \phi_{-\beta} \circ \mu_{e^{i\theta}}$. $\square$

So we've proved that

$$(III.C.6) \quad \text{Aut}(\Delta) \cong \left\{ e^{i\theta} \frac{z - \beta}{1 - \bar{\beta}z} \mid \beta \in \Delta, \theta \in \mathbb{R} \right\}.$$

An exercise in Chapter I.G showed that *amongst fractional linear transformations* the ones giving automorphisms of the upper half plane $\mathfrak{H}$ were exactly those of the form $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. In fact, those are all of the automorphisms, period:

III.C.7. THEOREM. $\text{Aut}(\mathfrak{H}) \cong \{f_M \mid M \in \text{SL}_2(\mathbb{R})\} (\cong \text{PSL}_2(\mathbb{R}))$.

PROOF. Let $g \in \text{Aut}(\mathfrak{H})$, so $x + iy := g(i) \in \mathfrak{H} \implies y > 0$. Write

$$M = \begin{pmatrix} \frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}} \\ 0 & \sqrt{y} \end{pmatrix} \in \text{SL}_2(\mathbb{R});$$

then

$$f_M(g(i)) = \frac{(\frac{1}{\sqrt{y}})(x + iy) - \frac{x}{\sqrt{y}}}{\sqrt{y}} = \frac{x + iy}{y} - \frac{x}{y} = \frac{iy}{y} = i.$$

Set

$$h := f_M \circ g: \mathfrak{H} \xrightarrow{\cong} \mathfrak{H}$$

$$i \mapsto i$$

which is an isomorphism because both f_M and g are. Let

$$F(z) := \frac{z - i}{z + i} = f_N, \quad N = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad N^{-1} = \frac{1}{2i} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}.$$

Then $f_N = F: i \mapsto 0$ and $f_{N^{-1}} = F^{-1}: 0 \mapsto i$, so

$$F \circ h \circ F^{-1}: \Delta \xrightarrow{\cong} \Delta$$

$$\text{sends } 0 \mapsto 0$$

hence must equal $\mu_{e^{i\theta}}$ (by Schwarz; see the beginning of this section). Writing $Q := \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$ ($\implies f_Q = \mu_{e^{i\theta}}$), we have therefore

$$h = F^{-1} \circ \mu_{e^{i\theta}} \circ F = f_{N^{-1}} \circ f_Q \circ f_N = f_{N^{-1}QN}.$$

Finally, put

$$M' := N^{-1}QN = \begin{pmatrix} \cos(-\frac{\theta}{2}) & -\sin(-\frac{\theta}{2}) \\ \sin(-\frac{\theta}{2}) & \cos(-\frac{\theta}{2}) \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R});$$

then

$$g = f_M^{-1} \circ h = f_{M^{-1}} \circ f_{M'} = f_{M^{-1}M'},$$

with $M^{-1}M' \in \mathrm{SL}_2(\mathbb{R})$. □

§ Pick's refinement and the Poincaré metric.

We turn to a generalization of the Schwarz lemma beyond the case where f sends $0 \mapsto 0$.

III.C.8. SCHWARZ-PICK LEMMA. *Let $f: \Delta \rightarrow \Delta$ be holomorphic, and $z_1, z_2 \in \Delta$ distinct. Then*

$$(III.C.9) \quad \left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)}f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z_1}z_2} \right|$$

and

$$(III.C.10) \quad |f'(z_1)| \leq \frac{1 - |f(z_1)|^2}{1 - |z_1|^2}.$$

If for any z_1, z_2 one gets equality in (III.C.9) or (III.C.10), then $f \in \mathrm{Aut}(\Delta)$.

PROOF. Using the disk automorphisms ϕ_α , we have that

$$F := \phi_{f(z_1)} \circ f \circ \phi_{-z_1} \in \mathcal{Hol}(\Delta)$$

sends $\Delta \rightarrow \Delta$ and $0 \rightarrow 0$. (Why?) Applying Schwarz (i) to F gives

$$|F(z)| \leq |z|,$$

and substituting $z = \phi_{z_1}(z_2)$ yields

$$|\phi_{f(z_1)}(f(z_2))| \leq |\phi_{z_1}(z_2)|,$$

which is (III.C.9). Moreover, noting that

$$\phi'_\alpha(z) = \frac{(1 - \bar{\alpha}z) + \bar{\alpha}(z - \alpha)}{(1 - \bar{\alpha}z)^2} = \frac{1 - |\alpha|^2}{(1 - \bar{\alpha}z)^2},$$

we can apply Schwarz (ii) to obtain

$$\begin{aligned} 1 \geq |F'(0)| &= |\phi'_{f(z_1)}(f(z_1))| |f'(z_1)| |\phi'_{-z_1}(0)| \\ &= \frac{1}{1 - |f(z_1)|^2} |f'(z_1)| (1 - |z_1|^2), \end{aligned}$$

which is (III.C.10).

Finally, equality in (III.C.9) [resp. (III.C.10)] evidently implies that $|F(z_2)| = |z_2|$ [resp. $|F'(0)| = 1$]. By Schwarz (iii) [resp. (iv)] this forces $F = \mu_{e^{i\theta}}$ hence $f = \phi_{-f(z_1)} \circ \mu_{e^{i\theta}} \circ \phi_{z_1}$. As a composition of automorphisms of Δ , this makes f an automorphism of Δ . $\square$

The statement above certainly looks rather technical and unmotivated. It also isn't how Pick originally stated it in 1916, which requires introducing a bit of differential geometry but is far more intuitive. In fact, after seeing this version you may prefer it to the Schwarz lemma.

III.C.11. DEFINITION. Let $U \subseteq \mathbb{C}$ be a region.

(i) A **conformal metric** on U is a non-negative function $\rho \in C^\infty(U)$, or more precisely the expression $\rho(z)|dz|$.

(ii) For $z \in U$ and $\xi \in \mathbb{C}$ (which we think of as a vector), the **length of ξ at z (with respect to ρ)** is

$$\|\xi\|_{\rho,z} := \rho(z)|\xi|.$$

(iii) If $\gamma: [a, b] \rightarrow U$ is a $\mathcal{C}^1$ path, the **length of γ (w.r.t. ρ)** is

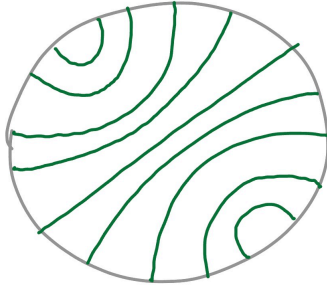
$$L_\rho(\gamma) := \int_a^b \|\gamma'(t)\|_{\rho, \gamma(t)} dt$$

III.C.12. EXAMPLE. Let $U = \Delta$, the unit disk centered at 0. The **Poincaré metric** is defined by

$$\rho(z) = \frac{1}{1 - z\bar{z}}.$$

The basic remarks here are:

- (Δ, ρ) is a complete metric space (with the distance between two points given by **geodesic distance** $= L_\rho$ of the “shortest” path) called the **hyperbolic disk**.
- In a metric space, **geodesics** are paths which everywhere locally are distance-minimizing; here they are the circular arcs (and line segments) which intersect $|z| = 1$ at right angles.



- Historically this was the first example of a non-Euclidean geometry (Gauss, Bolyai, Lobachevsky).

III.C.13. DEFINITION. Given two regions equipped with metrics (U, ρ) , (V, ψ) , a holomorphic map $f: U \rightarrow V$ is **isometric** (or an **isometry**) if

$$f^*(\psi(w)|dw|) = \rho(z)|dz|,$$

which explicitly means that $\psi(f(z))|f'(z)| = \rho(z)$.

III.C.14. THEOREM. Let $f: \Delta \rightarrow \Delta$ be holomorphic, and ρ be the Poincaré metric on Δ . Then:

- (i) f is distance decreasing (with respect to ρ), i.e.

$$(III.C.15) \quad (\rho \circ f) \cdot |f'| \leq \rho; \quad \text{and}$$

- (ii) equality in (III.C.15) for at least one point $\iff$ equality everywhere (i.e., f is an isometry) $\iff f \in \text{Aut}(\Delta)$.

PROOF. (i) We want to show

$$(\rho \circ f)|f'| \left(= \frac{|f'(z)|}{1 - |f(z)|^2} \right) \leq \frac{1}{1 - |z|^2}$$

But Schwarz-Pick (III.C.10) is $|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$.

(ii) Equality in (III.C.15) at a point means equality in (III.C.10) at a point, whence $f \in \text{Aut}(\Delta)$ by Schwarz-Pick.

It remains to show that f an automorphism implies f an isometry. Since every holomorphic automorphism of Δ can be written as a composition of $\mu_{e^{i\theta}}$'s and ϕ_α 's, and compositions of isometries are isometries, it suffices to check:

- $\mu_{e^{i\theta}}$ is an isometry: $\rho \circ \mu_{e^{i\theta}} = \rho$, and $|\mu'_{e^{i\theta}}(z)| \equiv |e^{i\theta}| = 1$.
- ϕ_α is an isometry: recalling that $\phi'_\alpha(z) = \frac{1 - \alpha\bar{\alpha}}{(1 - \bar{\alpha}z)^2}$,

$$\begin{aligned} (\rho \circ \phi_\alpha)(z) \cdot |\phi'_\alpha(z)| &= \frac{1}{1 - \left| \frac{z - \alpha}{1 - \bar{\alpha}z} \right|^2} \cdot \frac{1 - |\alpha|^2}{|1 - \bar{\alpha}z|^2} \\ &= \frac{1 - |\alpha|^2}{(1 - \bar{\alpha}z)(1 - \alpha\bar{z}) - (z - \alpha)(\bar{z} - \bar{\alpha})} \\ &= \frac{1 - |\alpha|^2}{1 - \bar{\alpha}z - \alpha\bar{z} + |\alpha z|^2 - |\alpha|^2 - \bar{\alpha}z - \alpha\bar{z} - |z|^2} \\ &= \frac{1 - |\alpha|^2}{(1 - |\alpha|^2)(1 - |z|^2)} \\ &= \frac{1}{1 - |z|^2} = \rho(z). \end{aligned} \quad \square$$

If you like differential geometry, you're probably already sold on this. But if not, why should we care about this reformulation, beyond the fact that it simplifies the expression of Schwarz-Pick and makes it more conceptual?

III.C.16. COROLLARY. *If the closure $\overline{f(\Delta)}$ belongs to Δ , then f has a unique fixed point p , and this p is the intersection of the sets $V_i = \underbrace{(f \circ \cdots \circ f)}_{i \text{ times}}(K)$ for any sufficiently large compact subset $K \subset \Delta$.*

SKETCH. Let $\epsilon = \frac{1}{2} d(\overline{f(\Delta)}, \Delta^c)$, which is positive since the two sets are disjoint and closed, and one of them is compact. Fix $z_0 \in \Delta$; then

$$g(z) := f(z) + \epsilon(f(z) - f(z_0))$$

maps Δ to Δ (why?). Theorem III.C.14(i) now implies

$$(\rho \circ g)(z) \cdot |g'(z)| \leq \rho(z) \quad (\forall z \in \Delta),$$

and taking $z = z_0$ gives

$$(\rho \circ f)(z_0) \cdot |g'(z_0)| \leq \rho(z_0).$$

Since $g'(z) = (1 + \epsilon)f'(z)$, this becomes

$$(III.C.17) \quad (\rho \circ f)(z_0) \cdot |f'(z_0)| \leq \frac{1}{1 + \epsilon} \rho(z_0).$$

This holds for any $z_0 \in \Delta$, since z_0 was arbitrary.

Integrating (III.C.17) over geodesics connecting points $\alpha, \beta \in \Delta$ yields

$$d_\rho(f(\alpha), f(\beta)) \leq \frac{1}{1 + \epsilon} d_\rho(\alpha, \beta),$$

and so f is a *contraction mapping* of the complete metric space (Δ, ρ) , at which point the result follows from the contraction-mapping fixed point theorem. $\square$

There is a powerful generalization of Theorem III.C.14 due to Ahlfors, which says that holomorphic maps between any two hyperbolic spaces have the distance-decreasing property. This has all kinds of applications to families of algebraic varieties, mapping class groups, and moduli spaces.

Exercises

- (1) Let K denote the subset of elements $M \in \mathrm{SL}_2(\mathbb{R})$ such that the fractional linear transformation f_M satisfies $f_M(\mathbf{i}) = \mathbf{i}$. Show that if $M \in K$, there exists a real θ such that $M = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$.
- (2) Suppose f is a 1-to-1 conformal mapping of the unit disk D onto a square Q with center at 0, and $f(0) = 0$. Prove that $f(\mathbf{i}z) = \mathbf{i}f(z)$.

- (3) If $f(z)$ is analytic and $\operatorname{Im}(f(z)) \geq 0$ for $\operatorname{Im}(z) > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \leq \frac{|z - z_0|}{|z - \overline{z_0}|} \quad \text{and} \quad \frac{|f'(z)|}{\operatorname{Im}(f(z))} \leq \frac{1}{y}$$

(where $z = x + iy$); show that equality in either expression means that f is a fractional linear transformation.

- (4) Prove that the path of the smallest Poincaré length joining two given points a, b in the unit disk Δ is on a circular arc which is perpendicular to the unit circle. [Hint: use a fractional linear transformation carrying one end point to the origin, and the other to a point on the positive real axis.] Show that this “geodesic distance” is given by

$$d_\rho(a, b) = \frac{1}{2} \log \left(\frac{1 + \left| \frac{a-b}{1-\bar{a}b} \right|}{1 - \left| \frac{a-b}{1-\bar{a}b} \right|} \right).$$