

III.D. Function series

§ Uniform limits of analytic functions.

Let $U \subset \mathbb{C}$ be an open set.

III.D.1. PROPOSITION. *Let $\{f_n : U \rightarrow \mathbb{C}\}$ converge uniformly on compact subsets of U , and write*

$$f(z) := \lim_{n \rightarrow \infty} f_n(z)$$

for the pointwise limit. Then

$$f_n \in \begin{cases} \mathcal{C}^0(U) \\ \mathcal{Hol}(U) \end{cases} \quad (\forall n) \quad \implies \quad f \in \begin{cases} \mathcal{C}^0(U) \\ \mathcal{Hol}(U) \end{cases}.$$

In the holomorphic case, the $\{f'_n(z)\}$ also converge uniformly on compact sets, to $f'(z)$.

PROOF. The $\mathcal{C}^0(U)$ case is the earlier result I.D.4, so assume that $f_n \in \mathcal{Hol}(U)$ ($\forall n$). Given $z_0 \in U$, and $0 < R < d(z_0, U^c)$, $\{f_n\}$ converges uniformly on $\bar{D}(z_0, R)$. For $z \in \bar{D}(z_0, R/2)$,

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial \bar{D}(z_0, R)} \frac{f_n(w)}{w - z} dw$$

by the local Cauchy formula (since f_n is holomorphic). So for $z \in \bar{D}(z_0, R/2)$,

$$\begin{aligned} \left| f_n(z) - \frac{1}{2\pi i} \int_{\partial \bar{D}(z_0, R)} \frac{f(w)}{w - z} dw \right| &= \frac{1}{2\pi} \left| \int_{\partial \bar{D}(z_0, R)} \frac{f_n(w) - f(w)}{w - z} dw \right| \\ &\leq \frac{1}{2\pi} 2\pi R \left\| \frac{f_n(w) - f(w)}{w - z} \right\|_{\partial \bar{D}(z_0, R) \times \bar{D}(z_0, R/2)} \\ &\leq \frac{R}{R/2} \|f_n - f\|_{\partial \bar{D}(z_0, R)} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

by uniform convergence on $\partial \bar{D}(z_0, R)$. Hence

$$(III.D.2) \quad f(z) \left(:= \lim_{n \rightarrow \infty} f_n(z) \right) = \frac{1}{2\pi i} \int_{\partial \bar{D}(z_0, R)} \frac{f(w)}{w - z} dw$$

for $z \in \bar{D}(z_0, R/2)$.

By the result II.E.12 on functions given by integrals over paths, f must be holomorphic there, with

$$f'(z) = \frac{1}{2\pi i} \int_{\partial D(z_0, R)} \frac{f(w)}{(w-z)^2} dw.$$

Since $f'_n(z) = \frac{1}{2\pi i} \int_{\partial D(z_0, R)} \frac{f_n(w)}{(w-z)^2} dw$, we have

$$\begin{aligned} \|f'_n(z) - f'(z)\|_{\bar{D}(z_0, R/2)} &= \frac{1}{2\pi} \left\| \int_{\partial \bar{D}(z_0, R)} \frac{f_n(w) - f(w)}{(w-z)^2} dw \right\|_{\bar{D}(z_0, R/2)} \\ &\leq R \left\| \frac{f_n(w) - f(w)}{(w-z)^2} \right\|_{\partial \bar{D}(z_0, R) \times \bar{D}(z_0, R/2)} \\ &\leq \frac{R}{R^2/4} \|f_n(w) - f(w)\|_{\partial \bar{D}(z_0, R)} \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

Finally, any compact set can be covered⁸ by a union of balls $D(z_i, R_i)$ with $R_i < d(z_i, U^c)$, hence by the $\bar{D}(z_i, R_i) \subset U$. $\square$

III.D.3. REMARK. (i) The first half of the proof could have been done more simply with Morera's theorem (Exercise (1) below). But we needed the representation (III.D.2) to apply II.E.12 for the second half.

(ii) If $\bigcup_{\alpha} U_{\alpha} = U$ (for a collection of open $U_{\alpha} \subset U$), then any compact $K \subset U$ is contained in a *finite* union of U_{α} 's. Hence it is enough to show uniform convergence on each U_{α} in order to satisfy the condition of the proposition.

(iii) Alternatively, we can start with $f_n \in \mathcal{H}ol(U_n)$,

$$U_1 \subset U_2 \subset \cdots \subset U = \bigcup_n U_n.$$

Since any compact $K \subset U$ is contained in some U_N , the hypothesis still makes sense (and the conclusion still holds for the pointwise limit f).

⁸Writing $d(K, U^c) > \epsilon > 0$, we can take all $R_i = \epsilon$, for example.

III.D.4. EXAMPLE (using (iii)). Consider

$$f_n(z) = \frac{z}{2z^n + 1} \in \mathcal{H}\mathcal{O}\mathcal{L}(D_{1/\sqrt[n]{2}}) \quad \text{and} \quad D_1 = \bigcup_n D_{1/\sqrt[n]{2}}.$$

The pointwise limit on D_1 (using $|z| < 1$) is

$$\lim_{n \rightarrow \infty} f_n(z) = \lim_{n \rightarrow \infty} \frac{z}{2z^n + 1} = \frac{z}{1} = z,$$

and

$$f_n(z) - z = \frac{z - 2z^{n+1} - z}{2z^n + 1} = \frac{-2z^{n+1}}{2z^n + 1} \rightarrow 0$$

uniformly on any $\bar{D}_{r < 1}$ (bound by $\frac{2r^{n+1}}{1-2r^n}$), and any compact $K \subset D_1$ is in such a $\bar{D}_r$. So by the Proposition, we reach the very exciting conclusion that $(f(z) =) z$ is holomorphic on D_1 .

III.D.5. EXAMPLE (using (ii)). For $k \in \mathbb{N}$, a **Dirichlet character mod k** is a function

$$\chi : \mathbb{N} \rightarrow \mathbb{C}$$

satisfying

- $\chi(1) = 1$,
- $\chi(n) = 0$ if $\gcd(n, k) \neq 1$,
- $\chi(n+k) = \chi(n)$, and
- $\chi(nm) = \chi(n)\chi(m)$.

For instance, we can take $\chi(n) =$

$$(k=4) \quad 0, 1, 0, -1, 0, 1, 0, -1, \dots$$

$$(k=5) \quad 0, 1, \mathbf{i}, -\mathbf{i}, -1, 0, 1, \mathbf{i}, -\mathbf{i}, -1, \dots$$

The corresponding **L-function** $L(\chi, z)$ is defined by a series

$$(III.D.6) \quad f(z) := \sum_{m \geq 1} F_m(z) := \sum_{m \geq 1} \frac{\chi(m)}{m^z}$$

for $\operatorname{Re}(z) > 1$. (Recall that $m^z = e^{z \log m}$.) These functions are immensely important in number theory. Indeed, when $\chi \equiv 1$ is the “trivial” character, (III.D.6) is the Riemann zeta function

$$\zeta(z) := \sum_{n \geq 1} \frac{1}{n^z}.$$

Now let

$$f_n(z) = \sum_{m=1}^n F_m(z),$$

which is analytic for $\operatorname{Re}(z) > 1$ (in fact, for $z \in \mathbb{C}$). Taking any $c > 1$, for $\operatorname{Re}(z) \geq c$ we have

$$\begin{aligned} |m^{-z}| &= |e^{-z \log m}| = |e^{-\operatorname{Re}(z) \log m}| |e^{-i \operatorname{Im}(z) \log m}| \stackrel{1}{=} e^{-\operatorname{Re}(z) \log m} \\ &\leq e^{-c \log m} = m^{-c}. \end{aligned}$$

Moreover, $\sum_{m \geq 1} \frac{1}{m^c}$ converges (since $c > 1$) by the integral test, so the Weierstrass M -test⁹ $\implies f_n(z)$ converges uniformly on

$$U_c := \{z \in \mathbb{C} \mid \operatorname{Re}(z) > c\}.$$

Now $\bigcup_{c>1} U_c = U_1$, and so the Proposition applies (via (i)) and $f(z)$ is a holomorphic function on U_1 .

One final application is

III.D.7. HURWITZ'S THEOREM. *Let U be a region and $f_n \in \mathcal{H}\mathcal{O}(U)$, with $f_n \rightarrow f$ uniformly on compact subsets. If $(\forall n \geq N)$ f_n is never 0 on U , then either $f \equiv 0$ or f is never 0 on U .*

PROOF. Assume $f \not\equiv 0$. Given any $z_0 \in U$, there exists $\epsilon > 0$ such that $f(z) \neq 0$ for $z \in \bar{D}^*(z_0, \epsilon) (\subset U)$. (This is because zeroes of holomorphic functions are isolated.) Write $\partial D = \partial D(z_0, \epsilon)$ and $\mu = \min_{z \in \partial D} |f(z)|$; then for $n \gg 0$,

$$\left\| \frac{1}{f_n} - \frac{1}{f} \right\|_{\partial D} = \left\| \frac{f - f_n}{f_n f} \right\|_{\partial D} \leq \frac{\|f - f_n\|_{\partial D}}{\frac{1}{2}\mu^2} \xrightarrow{n \rightarrow \infty} 0.$$

By the Proposition, $\|f'_n - f'\|_{\partial D} \rightarrow 0$, so the number of zeroes of f enclosed by ∂D is

$$\mathfrak{N}(f, \partial D) = \frac{1}{2\pi i} \oint_{\partial D} \frac{f'}{f} dz = \lim_{n \rightarrow \infty} \frac{1}{2\pi i} \oint_{\partial D} \frac{f'_n}{f_n} dz = \lim_{n \rightarrow \infty} \mathfrak{N}(f_n, \partial D) = 0.$$

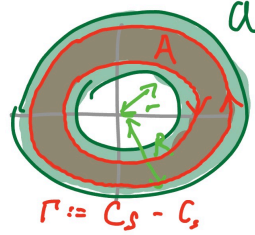
In particular, f has no zero at z_0 . □

⁹It turns out that $\chi(m)$ is always either 0 or a root of 1, of order $= \varphi(k)$, where φ is the Euler phi-function. Hence $\frac{1}{m^c} \geq \left| \frac{\chi(m)}{m^z} \right|$.

§ Laurent series.

We know how to write series representing holomorphic functions on a disk. What if they're not holomorphic at the center — or more generally, if $f \in \mathcal{H}ol(\mathfrak{a})$ where $\mathfrak{a}$ is the annulus

$$\mathfrak{a} = \{z \in \mathbb{C} \mid r < |z| < R\}?$$



If we pick a radius $\mathfrak{r}$ between r and R , and let $\gamma = \partial D_{\mathfrak{r}}$, then

$$g(z) := \int_{\gamma} \frac{f(w)}{w - z} dw$$

is holomorphic on $D_{\mathfrak{r}}$. Unless f actually had holomorphic extension to D_R , this *isn't the right function!*

Instead set $s := r + \epsilon$, $S := r - \epsilon$,

$$A := \{z \in \mathbb{C} \mid s < |z| < S\}.$$

For $z \in A$, the (homology version of) Cauchy's integral formula gives for $\Gamma := C_S - C_s (= \partial A \equiv_{\text{hom}} 0 \text{ in } A)$

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{w - z} dw \quad (\text{since } W(\Gamma, z) = 1) \\ &= \frac{1}{2\pi i} \int_{C_S} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{C_s} \frac{f(w)}{w - z} dw \\ &= \frac{1}{2\pi i} \int_{C_S} \sum_{n \geq 0} \frac{f(w) z^n}{w^{n+1}} dw + \frac{1}{2\pi i} \int_{C_s} \sum_{n \geq 0} \frac{f(w) w^n}{z^{n+1}} dw \\ &= \sum_{n \geq 0} \underbrace{\left(\frac{1}{2\pi i} \int_{C_S} \frac{f(w)}{w^{n+1}} dw \right)}_{=: a_n} z^n + \sum_{n \geq 0} \underbrace{\left(\frac{1}{2\pi i} \int_{C_s} w^n f(w) dw \right)}_{=: a_{-(n+1)}} z^{-(n+1)} \\ &= \sum_{m \in \mathbb{Z}} a_m z^m. \end{aligned}$$

The uniform convergence that lets us swap sums and integrals follows from the fact that for $w \in C_S$ [resp. C_s], $|\frac{z}{w}| < 1$ [resp. $|\frac{w}{z}| < 1$].

Taking $\epsilon \rightarrow 0$, we see that the representation is valid on all of $\mathfrak{A}$. This argument establishes the existence of Laurent expansions, but one doesn't typically use the formulas shown to compute the $\{a_m\}$.

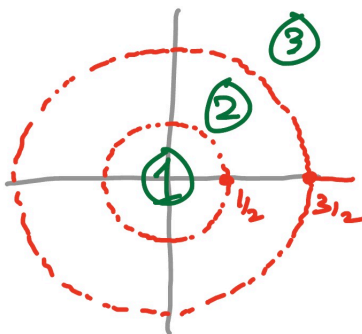
III.D.8. REMARK. (i) The above computation exhibits $f \in \mathcal{H}ol(\mathfrak{A})$ as the sum of $f_1 \in \mathcal{H}ol(D_R)$ and $f_2 \in \mathcal{H}ol(D_r^c)$.

(ii) Uniqueness of Laurent expansions: if $\sum_{m \in \mathbb{Z}} a_m z^m$ converges in $\mathfrak{A}$ to zero, then the above formulas for the $\{a_m\}$ do recover them, and obviously yield zero ($\forall n$).

III.D.9. EXAMPLE. Consider

$$f(z) = \frac{1}{(z - \frac{1}{2})(z - \frac{3}{2})} = \frac{1}{z - \frac{3}{2}} - \frac{1}{z - \frac{1}{2}}.$$

For the restriction of f to each of the three regions



we obtain a different Laurent expansion, by expanding each of

$$\begin{aligned} (1) \quad f(z) &= 2 \frac{1}{1-2z} - \frac{2}{3} \frac{1}{1-\frac{2}{3}z} \\ (2) \quad f(z) &= -\frac{1}{z} \frac{1}{1-\frac{1}{2z}} - \frac{2}{3} \frac{1}{1-\frac{3}{2}z} \\ (3) \quad f(z) &= -\frac{1}{z} \frac{1}{1-\frac{1}{2z}} + \frac{1}{z} \frac{1}{1-\frac{3}{2z}} \end{aligned}$$

in geometric series. For instance, in (2) we have $\frac{1}{2} < |z| < \frac{3}{2}$, which translates to $|\frac{1}{2z}| < 1$ and $|\frac{3}{2}z| < 1$. Also note that (1) is just a power-series expansion.

III.D.10. EXAMPLE. $\cos\left(\frac{1}{z}\right) = \sum_{m \geq 0} \frac{(-1)^m}{(2m)!} \left(\frac{1}{z}\right)^{2m}$ on $\mathbb{C}^*$, by plugging $\frac{1}{z}$ into the power series for $\cos$.

III.D.11. EXAMPLE. Recall that the Bernoulli numbers B_n are defined by

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} \frac{B_n}{n!} z^n = 1 - \frac{z}{2} + \sum_{m \geq 1} \frac{B_{2m}}{(2m)!} z^{2m}.$$

Combined with

$$\begin{aligned} \frac{2iz}{e^{2iz} - 1} &= z \cdot \frac{2i}{e^{iz}(e^{iz} - e^{-iz})} \\ &= z \frac{e^{-iz}}{\sin z} \\ &= z \frac{\cos z - i \sin z}{\sin z} \\ &= z \cot z - iz, \end{aligned}$$

this implies

$$\begin{aligned} z \cot z &= iz + 1 - \frac{2iz}{2} + \sum_{m \geq 1} \frac{B_{2m}}{(2m)!} (2iz)^{2m} \\ &= 1 + \sum_{m \geq 1} \frac{B_{2m}(-4)^m}{(2m)!} z^{2m}, \end{aligned}$$

which finally gives

$$(III.D.12) \quad \cot z = \frac{1}{z} + \sum_{m \geq 1} \frac{B_{2m}(-4)^m}{(2m)!} z^{2m-1},$$

valid on $0 < |z| < \pi$.

Exercises

- (1) Give a shorter proof of the implication $f_n \in \mathcal{H}ol(U) \ (\forall n) \implies f \in \mathcal{H}ol(U)$ in Proposition III.D.1 using Morera's theorem.
- (2) Weierstrass's theorem for a real interval $[a, b]$ says that a continuous function can be uniformly approximated by polynomials.

Does this still hold for the closed unit disk, i.e. can every continuous function on the disk $\overline{D}_1$ be uniformly approximated by polynomials in z ?

- (3) Let $\{a_n\}$ be a sequence of complex numbers. Show that the series $\sum_n \frac{a_n}{n^s}$, if it converges absolutely for some complex s , converges absolutely in a right half-plane $\operatorname{Re}(s) > \sigma_0$, and uniformly in $\operatorname{Re}(s) > \sigma_0 + \epsilon$ for every $\epsilon > 0$. Show that the series defines an analytic function in this half-plane. Represent its derivative in series form.

- (4) Find the Laurent expansion of $f(z) = \frac{1}{(z-1)^2(z+1)^2}$ for $|z| > 1$.

- (5) Let $z_0 \in U$ (open set), and $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$ with $f'(z_0) \neq 0$. Show that

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)},$$

where C is a small circle centered at z_0 . [Hint: what is the Laurent series of $\frac{1}{f(z) - f(z_0)}$ in a punctured disk about z_0 ?]

- (6) Problem (2) of Chapter III.A had you find the coefficients of degree ≤ 7 in the power-series expansion of $\tan(z)$ about $z = 0$. Now find a *general formula* for the coefficients by using the expansion of $z \cot(z)$ from Example III.D.11.