

III.E. Isolated Singularities

§ Three types of singularities.

Let $f \in \mathcal{H}ol(U)$, $U \subset \mathbb{C}$ open. A **singularity** of f is really just a point (of U^c) where f is not defined.

III.E.1. DEFINITION. $p \in U^c$ is an **isolated singularity** of f if there is an $R > 0$ such that $\overline{D}^*(p, R) \subset U$. (That is, p is not an accumulation point of the other singularities of f .)

Consider the case $p = 0$: let $f \in \mathcal{H}ol(\overline{D}_R^*)$; then $f(z) = \sum_{n \in \mathbb{Z}} a_n z^n$ there, for some unique $\{a_n\}_{n \in \mathbb{Z}}$. The possibilities for f 's isolated singularity at 0 comprise three different “species”:

- **removable singularity**: all $a_n = 0$ for $n < 0$; equivalently, we can extend f to $\mathcal{H}ol(\overline{D}_R)$.
- **pole** of order m : all $a_n = 0$ for all $n < -m$, but $a_{-m} \neq 0$; here we write $\text{ord}_0(f) = -m$. A pole of order 1 is **simple**.
- **essential singularity**: $a_n \neq 0$ for infinitely many $n < 0$.

III.E.2. PROPOSITION (Riemann). *0 is a removable singularity if and only if f is bounded on $\overline{D}_R^*$.*

PROOF. Assuming f is locally bounded, with $\|f\|_{\overline{D}_R^*} =: B$, the formula for Laurent coefficients yields

$$|a_{-(n+1)}| = \left| \frac{1}{2\pi i} \int_{C_\epsilon} w^n f(w) dw \right| \leq \frac{2\pi\epsilon}{2\pi} \epsilon^n \|f\|_{C_\epsilon} \leq \epsilon^{n+1} B \xrightarrow{\epsilon \rightarrow 0} 0$$

for all $n \geq 0$. The converse direction is obvious. $\square$

The following gives a different approach to Riemann's result, as well as a slightly stronger statement:

III.E.3. PROPOSITION. *0 is removable $\iff \lim_{z \rightarrow 0} z f(z) = 0$.*

PROOF. Using the “ f extends holomorphically across the puncture” form of the definition of removable singularity, we can forgo Laurent series entirely. Suppose $\lim_{z \rightarrow 0} z f(z) = 0$. Then for $z \in D_R^*$,

taking $0 < \epsilon < |z| < R \implies$

$$f(z) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i} \oint_{C_\epsilon} \frac{f(w)}{w-z} dw. \quad \nearrow 0 \text{ (by II.C.3)}$$

The first integral yields a holomorphic function inside D_R , extending f across the origin. $\square$

By means of Laurent series one readily verifies the following results on poles:

- $\text{ord}_0(f) = -m \iff z^m f(z) \in \mathcal{H}ol(D_R)$ and is nonzero at $z = 0$;
- $\text{ord}_0(f) + \text{ord}_0(g) = \text{ord}_0(fg)$, $\text{ord}_0(f) - \text{ord}_0(g) = \text{ord}_0(f/g)$;

and consequently,

- if $f \not\equiv 0$ is holomorphic with $f(0) = 0$, then $\frac{1}{f}$ has a pole at 0.

A more interesting result is the

III.E.4. THEOREM (Casorati-Weierstrass). *If 0 is an essential singularity of f , then*

$$(III.E.5) \quad f(D_\epsilon^*) \text{ is dense in } \mathbb{C} \quad (\forall \epsilon > 0).$$

III.E.6. REMARK. This means that if z_0 is an essential singularity of f , then $\lim_{z \rightarrow z_0} |f(z)|$ is *undefined*, even if we think of ∞ as a valid limit. If the limit is ∞ , then you've got a *pole*, not an essential singularity.

PROOF. We argue the contrapositive: if (III.E.5) is *not* the case, then there exists $\alpha \in \mathbb{C}$ such that $f(D_\epsilon^*)$ omits a neighborhood of α :

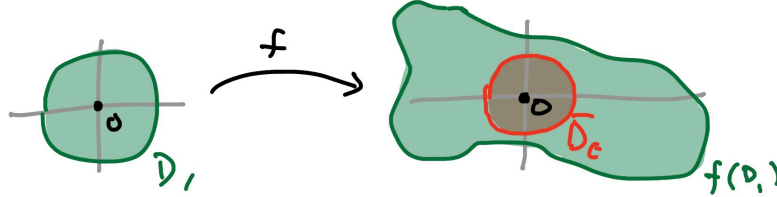
$$\begin{aligned} & |f(z) - \alpha| > \delta (> 0) \quad \forall 0 < |z| < \epsilon \\ \implies & \left| \frac{1}{f(z) - \alpha} \right| < \frac{1}{\delta} (< \infty) \quad \forall z \in D_\epsilon^* \\ \implies & \frac{1}{f(z) - \alpha} \in \mathcal{H}ol(D_\epsilon) \quad [\text{by Riemann}] \\ \implies & f(z) - \alpha \text{ has (at worst) a pole at } 0 \\ \implies & f(z) \text{ has (at worst) a pole at } 0 \end{aligned}$$

$\implies$ no essential singularity. $\square$

III.E.7. COROLLARY. *The only analytic automorphisms $f: \mathbb{C} \rightarrow \mathbb{C}$ are of the form $f(z) = az + b$ (translation-dilation-rotation).*

PROOF. We assume $f(0) = 0$, and attempt to show $f(z) = az$.

By the open mapping theorem, $f(D_1)$ is an open neighborhood of the origin,



so contains $\overline{D}_\epsilon$ for some $\epsilon > 0$. Taking $\zeta = \frac{1}{z}$,

$$\begin{aligned} 0 < |\zeta| < 1 &\implies |z| > 1 \implies f(z) \notin f(D_1) \quad [\text{since } f \text{ is invertible}] \\ &\implies |f(z)| > \epsilon \implies |f(\frac{1}{\zeta})| > \epsilon. \end{aligned}$$

Setting $h(\zeta) := f(\frac{1}{\zeta})$, this shows that $h(D_1^*)$ omits a neighborhood of 0. By Casorati-Weierstrass, h cannot have an essential singularity at $\zeta = 0$ — at worst, it can have a pole.

But now $f(z) = \sum_{n \geq 0} a_n z^n \implies h(\zeta) = f(\frac{1}{\zeta}) = \sum_{n \geq 0} a_n \zeta^{-n}$, and we have just shown these negative powers must terminate at some m . That is, $a_n = 0$ for $n > m$, which makes $f(z)$ a polynomial, which we can write as $a \prod_{i=1}^m (z - z_i)$. If the $\{z_i\}$ are not all equal, then f sends two different numbers to 0 and is not 1-to-1, a contradiction.

Thus $f(z) = a(z - z_0)^m$. If $m > 1$ then f is not 1-to-1 in a disk about 0, again a contradiction. So $m = 1$ and $f(z) = a(z - z_0)$. But if $f(0) = 0$, then $z_0 = 0$, and $f(z) = az$. $\square$

III.E.8. REMARK. The *Big Picard Theorem* is a substantial strengthening of Casorati-Weierstrass which we will prove later in this course. It says that *if f has an essential singularity, then on any D_ϵ^* , f takes all possible complex values, with at most a single exception, infinitely often.*

III.E.9. EXAMPLE. $e^{1/z}$ has an essential singularity at 0, and obviously omits the value 0.

We conclude this part with an application of Riemann's removable singularity result:

III.E.10. TAYLOR'S THEOREM. Given $f \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$ and $a \in U$, with $\overline{D}(a, r) \subset U$, we may write

$$f(z) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (z-a)^k + f_n(z)(z-a)^n$$

with $f_n \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$, $f^{(n)}(a) = n!f_n(a)$, and

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D(a, r)} \frac{f(w)}{(w-a)^n(w-z)} dw$$

for $|z-a| < r$.

PROOF. Since $g_1(z) := \frac{f(z)-f(a)}{z-a}$ times $(z-a)$ goes to 0 as $z \rightarrow a$, Prop. III.E.3 $\implies g_1 = f_1|_{U \setminus \{a\}}$ for $f_1 \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$. Repeat the process, extending $g_2(z) = \frac{f_1(z)-f_1(a)}{z-a}$ to $f_2 \in \mathcal{H}\mathcal{O}\mathcal{L}(U)$. This gives

$$\begin{aligned} f(z) &= f_0(a) + (z-a)f_1(z) \quad [\text{where } f_0(a) := f(a)] \\ &= f_0(a) + (z-a)f_1(a) + (z-a)^2f_2(z) = \dots \end{aligned}$$

$$\text{hence } 2\pi i f_n(z) = \int_{\partial D} \frac{f_n(w)}{w-z} dw = \int_{\partial D} \frac{f(w)dw}{(w-a)^n(w-z)} - \sum_{\ell=1}^n F_\ell(a)$$

where $F_\ell(a) = \int_{\partial D} \frac{f_{n-\ell}(a)dw}{(w-a)^\ell(w-z)}$ and $D = D(a, r)$. But the $F_\ell(a) = 0$ since $\int_{\partial D} \frac{dw}{(w-a)^\ell(w-z)} = \sum_{m \geq 0} \int_{\partial D} \frac{(z-a)^m dw}{(w-a)^{\ell+m+1}}$ and $\int_{\partial D} \frac{dw}{(w-a)^k} = 0$ for $k \geq 2$. $\square$

III.E.11. REMARK. Taylor theorem is used in [Ahlfors] to

(i) bound the error in the Taylor series partial sums:

$$|f_n(z)(z-a)^n| \leq \frac{1}{2\pi} 2\pi r \frac{\|f\|_{\partial D} |z-a|^n}{r^n(r-|z-a|)} \xrightarrow{n \rightarrow \infty} 0 \quad \text{for } \frac{|z-a|}{r} \leq 1-\epsilon,$$

which yields uniform convergence of the series on compact sets.

(ii) show that zeroes are isolated: a similar estimate shows that if all $f^{(k)}(a)$ vanish, then f vanishes on an open disk; thus the set $\mathcal{S}$ where all $f^{(k)}$ vanish is open. Obviously $\mathcal{S}^c$ is open. If U is a *connected* open set, either $\mathcal{S}$ or $\mathcal{S}^c$ is empty. If $f \not\equiv 0$, then $U \neq \mathcal{S}$, and so $U = \mathcal{S}^c$. Locally at any zero z_0 of f , some $f^{(k)}$ is therefore nonzero, and so $f \not\equiv 0$ in a punctured neighborhood of z_0 .

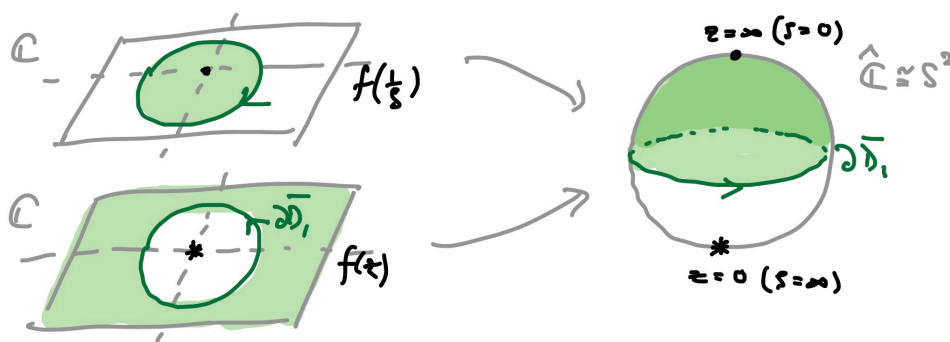
§ Meromorphic functions.

As above, let $U \subset \mathbb{C}$ be an open set. A subset $S \subset U$ is **discrete** if it has no limit points in U .

III.E.12. DEFINITION. f is a **meromorphic function** on U , written $f \in \text{Mer}(U)$, if f restricts to a holomorphic function on the complement of a discrete set $S \subset U$, with poles at each point of S .

III.E.13. EXAMPLE. If $g_1, g_2 \in \text{Hol}(U)$, with g_2 not identically zero, then $g_1/g_2 \in \text{Mer}(U)$.

Given a meromorphic or holomorphic function $f(z)$ on $\mathbb{C}$, we may inquire as to its behavior at ∞ . In fact, it is useful to consider it as a function (with singularities) on the entire Riemann sphere. To do this, write $\zeta = \frac{1}{z} \leftrightarrow z = \frac{1}{\zeta}$ and notice that f is completely described by $f(z)$ for $z \in \overline{D}_1$ and $f(\frac{1}{\zeta})$ for $\zeta \in \overline{D}_1$.



We say $f(z)$ has a singularity of a certain type at $z = \infty$ if $f(\frac{1}{\zeta})$ has this type of singularity at $\zeta = 0$. (The same goes for the order of a zero or pole at ∞ .)

Now given a cover of $\hat{\mathbb{C}}$ by open sets (which take the form of open sets in $\mathbb{C}$ but also $\{\infty\} \cup K^c$ for $K \subset \mathbb{C}$ compact), we get finite subcovers on each of $\overline{D}_{1,z}$ (lower hemisphere) and $\overline{D}_{1,\zeta}$ (upper hemisphere). Hence $\hat{\mathbb{C}}$ is compact.

III.E.14. REMARK. ∞ is an accumulation point of $\mathbb{Z}$. So for example the zeroes of $\cos(z)$ have ∞ as an accumulation point; this guarantees that it has ∞ as an essential singularity.

III.E.15. PROPOSITION. *If $f \in \mathcal{Mer}(\mathbb{C}) \setminus \{0\}$,¹⁰ then f has finitely many zeroes and poles in $\overline{D}_1$ (and more generally in any compact set).*

PROOF. If the polar set of f in $\overline{D}_{1+\epsilon}$ is infinite, then it has an accumulation point; but then it isn't discrete, violating Def. III.E.12. ✖ So this set is finite: the poles of f in $\overline{D}_{1+\epsilon}$ are $\{\alpha_1, \dots, \alpha_m\}$. Let $U = D_{1+\epsilon} \setminus \{\alpha_1, \dots, \alpha_m\}$.

If f has infinitely many zeroes in $\overline{D}_1$, then they have an accumulation point in $\overline{D}_1$ which *cannot* be one of the $\{\alpha_i\}$. (Near α_i , $|f|$ is big.) Hence, the zeroes have an accumulation point in U , and $f \equiv 0$, contradiction. $\square$

III.E.16. COROLLARY. $f \in \mathcal{Mer}(\hat{\mathbb{C}}) \setminus \{0\} \implies f$ has finitely many zeroes and poles.

PROOF. Writing $\zeta = \frac{1}{z}$, we have $\hat{\mathbb{C}} = \{|z| \leq 1\} \cup \{|\zeta| \leq 1\}$, and restriction gives $f(z) \in \mathcal{Mer}(\mathbb{C})$, $f(\frac{1}{\zeta}) \in \mathcal{Mer}(\mathbb{C})$. Now apply the Proposition. $\square$

III.E.17. THEOREM. $f \in \mathcal{Mer}(\hat{\mathbb{C}}) \implies f$ is a rational function (quotient of polynomials).

PROOF. If $f \equiv 0$ the result is trivial. Assume otherwise, and let $m := \text{ord}_\infty(f(z)) = \text{ord}_0(f(\frac{1}{\zeta}))$. For finitely many $\alpha_i \in \mathbb{C}$ we have $m_i := \text{ord}_{\alpha_i}(f) \in \mathbb{Z} \setminus \{0\}$. Hence

$$G(z) := \frac{f(z)}{\prod_i (z - \alpha_i)^{m_i}}$$

has no zeroes or poles on $\mathbb{C}$, making $G, \frac{1}{G}$ entire. Now $\text{ord}_\infty(G) = m + \sum m_i$. If this is ≥ 0 , then G is bounded (since $\hat{\mathbb{C}}$ is compact) and thus constant by Liouville. If $\text{ord}_\infty(G) < 0$, the same argument applies to $\frac{1}{G}$. Conclude that $\frac{f(z)}{\prod (z - \alpha_i)^{m_i}} = C$, and rewrite this as $f(z) = C \prod (z - \alpha_i)^{m_i} = \frac{P(z)}{Q(z)}$, where $P(z) = \prod_{i:m_i > 0} (z - \alpha_i)^{m_i}$ and $Q(z) = \prod_{i:m_i < 0} (z - \alpha_i)^{-m_i}$. $\square$

¹⁰The " $\setminus \{0\}$ " just means that we are excluding the zero-function so that f cannot be identically zero.

III.E.18. COROLLARY. If $f \in \mathcal{M}er(\mathbb{C})$ and $\lim_{|z| \rightarrow \infty} |f(z)| = \infty$, then f is rational.

PROOF. For $|z| > \frac{1}{\delta}$, $|f(z)| > \frac{1}{\epsilon}$; i.e., $|\zeta| < \delta$ gives $|f(\frac{1}{\zeta})| > \frac{1}{\epsilon}$. Now Casorati-Weierstrass $\implies f(\frac{1}{\zeta})$ doesn't have an essential singularity at $\zeta = 0 \implies$ it has a pole there. Apply Theorem III.E.17. $\square$

III.E.19. COROLLARY.

- (a) $f \in \mathcal{H}ol(\mathbb{C})$ with pole at $\infty \implies f$ polynomial.
- (b) $f \in \mathcal{H}ol(\hat{\mathbb{C}}) \implies f$ constant.

PROOF. (a) $f = \frac{P}{Q}$ by Theorem III.E.17; Q can't have zeroes on $\mathbb{C} \implies Q$ constant.

(b) by (a), $f = P = C \prod (z - a_i)^{k_i}$ ($k_i \geq 0$). $\implies \text{ord}_{\infty}(f) = -\sum k_i$. But $\text{ord}_{\infty}(f) \geq 0 \implies$ all $k_i = 0$. $\square$

It will turn out that all *simply connected* Riemann surfaces (complex 1-manifolds) are biholomorphic to $\hat{\mathbb{C}}$, $\mathbb{C}$, or D_1 .¹¹ If we know the automorphism groups of the latter, we know them for the Riemann surface. With this in mind:

- (1) $\text{Aut}(D_1) \cong \text{PSL}_2(\mathbb{R})$
- (2) $\text{Aut}(\mathbb{C}) \cong \mathbb{R}^* \ltimes \mathbb{R}$ (via transformations $az + b$)
- (3) $\text{Aut}(\hat{\mathbb{C}}) \cong \text{PSL}_2(\mathbb{C})$

PROOF OF (3). Since $\underline{\text{FLT}} \cong \text{PSL}_2(\mathbb{C})$, it suffices to show $\text{Aut}(\hat{\mathbb{C}}) = \underline{\text{FLT}}$. The $\supseteq$ inclusion is clear. Now let $f \in \text{Aut}(\hat{\mathbb{C}})$, i.e. f is a 1-to-1 holomorphic map $\hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, hence a meromorphic function with a single pole, which is simple. By the Corollary above, f is a rational function P/Q . Written in lowest terms, Q must have degree ≤ 1 . In order to be 1-to-1, also, $\deg(P) \leq 1$. So f is an FLT. $\square$

¹¹This is a bit of a red herring: the ones which are actually interesting topologically are *not* simply connected. But their universal covers are, and so this is saying that there are only three possibilities for the universal cover. We'll return to this idea.

Exercises

- (1) Is there a polynomial $P(z)$ such that $P(z)e^{1/z}$ is an entire function?
- (2) The Fibonacci numbers are defined by $c_0 = 0, c_1 = 1, c_n = c_{n-1} + c_{n-2}$. Show that the c_n are the Taylor coefficients of a rational function, and use this to determine a "closed expression" for c_n (as a sum of binomial coefficients).
- (3) Let f be meromorphic on $\mathbb{C}$ but not entire. Let $g(z) = e^{f(z)}$. Show that g is not meromorphic on $\mathbb{C}$.
- (4) Show that $\sum_{n \geq 1} \frac{1}{(z+n)^2}$ defines a meromorphic function on $\mathbb{C}$ and determine its set of poles (and their orders). [Hint: apply the Weierstrass M -test on compact subsets of $\mathbb{C} \setminus \mathbb{Z}_{\leq -1}$.]
- (5) The expression $\{f, z\} := \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$ is called the *Schwarzian derivative* of f .¹² If f has a multiple zero or pole (say $f(z) = a(z - z_0)^m + \dots$), show that the leading term in the Laurent expansion of $\{f, z\}$ is $\frac{1}{2}(1 - m^2)(z - z_0)^{-2}$.

¹²One interesting property of the operation $f \mapsto \{f, z\}$ is that it kills fractional linear transformations.