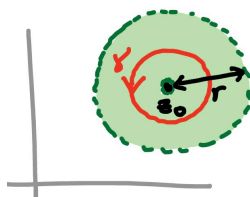


III.F. Residue calculus

§ Residues of functions.

Recall the setup for studying isolated singularities:



Any $f \in \mathcal{H}ol(D^*(z_0, r))$ may be expressed by Laurent series there:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$$

III.F.1. DEFINITION. $\text{Res}_{z_0}(f) := a_{-1}$

III.F.2. PROPOSITION. For γ a small counterclockwise circle about z_0 ,

$$\int_{\gamma} f(z) dz = 2\pi i \text{Res}_{z_0}(f).$$

PROOF. Since $\int_{\gamma} (z - z_0)^n dz = 0$ for $n \neq -1$,

$$\int_{\gamma} \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n dz = \sum_{n \in \mathbb{Z}} a_n \int_{\gamma} (z - z_0)^n dz = 2\pi i a_{-1}. \quad \square$$

III.F.3. REMARK. f has a primitive in $D^* \iff \text{Res}_{z_0}(f) = 0$.

Rules for computing residues

(0) Brute force with Laurent series:

III.F.4. EXAMPLE. $\text{Res}_0 \left(\frac{e^z}{\sin^2 z} \right) = 1$:

$$\begin{aligned} \frac{e^z}{\sin^2 z} &= \frac{1 + z + \frac{z^2}{2} + \cdots}{(z - z^3/6 + \cdots)^2} = \frac{1 + z + \frac{z^2}{2} + \cdots}{z^2(1 - \frac{z^2}{3} + \cdots)} \\ &= \left(\frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} + \cdots \right) \left(1 + \frac{z^2}{3} - \cdots \right) = \frac{1}{z^2} + \frac{1}{z} + \cdots \end{aligned}$$

III.F.5. EXAMPLE. $\text{Res}_2 \left(\frac{1}{z(z-2)^2} \right) = -\frac{1}{4}$:

$$\begin{aligned} \frac{1}{z(z-2)^2} &= \frac{1}{z-2+2} \cdot \frac{1}{(z-2)^2} = \frac{1}{2} \frac{1}{1 + \frac{z-2}{2}} \cdot \frac{1}{(z-2)^2} \\ &= \frac{\frac{1}{2} \left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \dots \right)}{(z-2)^2} = \frac{1/2}{(z-2)^2} - \frac{1/4}{z-2} + \dots \end{aligned}$$

There's a better method for the 2nd example.

In fact, here are six methods suited for various sorts of examples:

(1) f holomorphic at $z_0 \implies \text{Res}_{z_0} \left(\frac{f(z)}{z-z_0} \right) = f(z_0)$.

III.F.6. EXAMPLE. $\text{Res}_{\hbar} \left(\frac{\log(z)}{z-\hbar} \right) = \log \hbar$

PROOF OF (1). $f(z) = b_0 + b_1(z-z_0) + \dots$, where $b_0 = f(z_0)$.
Then $\frac{f(z)}{z-z_0} = \frac{b_0}{z-z_0} + b_1 + \dots$. $\square$

(2) f holomorphic at $z_0 \implies \text{Res}_{z_0} \left(\frac{f(z)}{(z-z_0)^n} \right) = \frac{f^{(n-1)}(z_0)}{(n-1)!}$.

III.F.7. EXAMPLE. $\text{Res}_0 \left(\frac{\sin z}{z^{10}} \right) = \frac{\left[\left(\frac{d}{dz} \right)^9 \sin z \right]_0}{9!} = \frac{\cos 0}{9!} = \frac{1}{9!}$

III.F.8. EXAMPLE. $\text{Res}_2 \left(\frac{1}{z(z-2)^2} \right) = \frac{-1/2^2}{1!} = -\frac{1}{4}$ (using $\frac{d}{dz} \frac{1}{z} = \frac{-1}{z^2}$)

PROOF OF (2). Using Taylor's theorem,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + \dots$$

hence $\frac{f(z)}{(z-z_0)^n} = \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!} \frac{1}{z-z_0} + \dots$ $\square$

(3) f has a simple pole at $z_0 \implies \text{Res}_{z_0}(f) = \lim_{z \rightarrow z_0} (z-z_0)f(z)$,
which we can compute via substitution or L'Hôpital.

III.F.9. REMARK. Recall that f has a *simple* pole at z_0 when the Laurent series starts at $n = -1$ and $a_{-1} \neq 0$.

III.F.10. EXAMPLE. $\text{Res}_i \left(\frac{e^{\pi z}}{z^2+1} \right) = \lim_{z \rightarrow i} \frac{(z-i)e^{\pi z}}{z^2+1} = \lim_{z \rightarrow i} \frac{e^{\pi z}}{z+i} = \frac{e^{\pi i}}{2i} = \frac{i}{2}$, and $\text{Res}_{-i} \left(\frac{e^{\pi z}}{z^2+1} \right) = \lim_{z \rightarrow -i} \frac{e^{\pi z}}{z-i} = \frac{e^{-\pi i}}{-2i} = -\frac{i}{2}$.

PROOF OF **(3)**. $f(z) = a_{-1}(z - z_0)^{-1} + \text{holo.} \implies (z - z_0)f(z) = a_{-1} + (z - z_0)(\text{holo.}) \rightarrow a_{-1}$. $\square$

Here is another generalization of (1):

(4) If f has a simple pole at z_0 and g is holomorphic there, then

$$\text{Res}_{z_0}(fg) = g(z_0)\text{Res}_{z_0}(f)$$

where we can compute $\text{Res}_{z_0}(f)$ by (3) or another method. [See next section for an example; the proof is an exercise.]

(5) f holomorphic at z_0 , $f(z_0) = 0$, $f'(z_0) \neq 0 \implies$

$$\text{Res}_{z_0}\left(\frac{1}{f}\right) = \frac{1}{f'(z_0)}.$$

III.F.11. EXAMPLE. $\text{Res}_0\left(\frac{1}{e^{3z}-1}\right) = \frac{1}{3}$, since

$$\frac{d}{dz}(e^{3z} - 1)\Big|_{z=0} = 3e^{3z}\Big|_{z=0} = 3.$$

PROOF OF **(5)**. Writing $f(z) = b_1(z - z_0)(1 + h(z))$ with $h(z_0) = 0$ and h locally holomorphic,

$$\frac{1}{f(z)} = \frac{1}{b_1(z - z_0)}(1 - h + h^2 - \dots) \implies "a_{-1}" = \frac{1}{b_1} = \frac{1}{f'(z_0)}.$$

$\square$

(6) For z_0 a zero or pole of f with $\text{ord}_{z_0}(f) = m$,

$$\text{Res}_{z_0}(f'/f) = m.$$

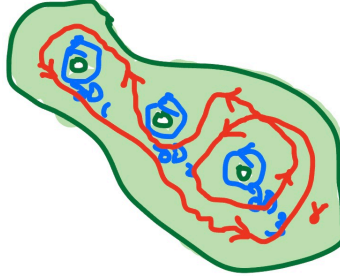
III.F.12. EXAMPLE. For ζ any 10th root of 1, $\text{Res}_0\left(\frac{10z^9}{z^{10}-1}\right) = 1$, since $z^{10} - 1$ has a simple (order 1) zero at ζ .

PROOF OF **(6)**. From $f(z) = a_m(z - z_0)^m(1 + h(z))$, with $h(z_0) = 0$ and h holomorphic,

$$\begin{aligned} f'(z) &= ma_m(z - z_0)^{m-1}(1 + h(z)) + a_m(z - z_0)^m h'(z) \\ \implies \frac{f'(z)}{f(z)} &= \frac{m}{z - z_0} + \frac{h'(z)}{1 + h(z)} \implies \text{Res}_{z_0}\left(\frac{f'}{f}\right) = m. \end{aligned} \quad \square$$

§ Residue formulas.

For each of the results that follow, $U \subseteq \mathbb{C}$ is open, and $\gamma \subset U$ is a closed path which is *homologous to zero in U* and avoids points where the integrand is singular.



III.F.13. RESIDUE THEOREM.

$$f \in \mathcal{H}ol(U \setminus \{z_1, \dots, z_m\}) \implies \oint_{\gamma} f dz = 2\pi i \sum_{j=1}^m W(\gamma, z_j) \text{Res}_{z_j}(f).$$

PROOF. We use $\gamma \equiv_{\text{hom}} \sum_j W(\gamma, z_j) \partial D_j \subset U \setminus \{z_1, \dots, z_m\}$, as shown in the picture, and $\int_{\partial D_j} f dz = 2\pi i \text{Res}_{z_j}(f)$. $\square$

III.F.14. EXAMPLE. Referring to III.F.10,

$$\begin{aligned} \oint_{|z|=2} \frac{e^{\pi z}}{1+z^2} dz &= 2\pi i \left(\text{Res}_i \left(\frac{e^{\pi z}}{1+z^2} \right) + \text{Res}_{-i} \left(\frac{e^{\pi z}}{1+z^2} \right) \right) \\ &= 2\pi i \left(\frac{i}{2} - \frac{i}{2} \right) = 0. \end{aligned}$$

III.F.15. ARGUMENT PRINCIPLE. If $f \in \mathcal{M}er(U)$, then γ can have nonzero winding number about only finitely many zeroes and poles of f . Calling these $z_1, \dots, z_m$, we have

$$\oint_{\gamma} \frac{f'}{f} dz = 2\pi i \sum_{j=1}^m W(\gamma, z_j) \text{ord}_{z_j}(f).$$

PROOF. The proof of the first statement is the same as in III.E.15. Now use (6) above and the Residue Theorem. $\square$

III.F.16. EXAMPLE. $\int_{|z|=2} \frac{10z^9}{z^{10}-1} dz = 2\pi i \sum_{j=1}^{10} 1 \cdot 1 = 20\pi i.$

III.F.17. GENERALIZED ARGUMENT PRINCIPLE. *With the same description of $z_1, \dots, z_m$ as in III.F.15,*

$$\begin{cases} f \in \text{Mer}(U) \\ g \in \text{Hol}(U) \end{cases} \implies \oint_{\gamma} \frac{f'}{f} g \, dz = 2\pi i \sum_j W(\gamma, z_j) \text{ord}_{z_j}(f) g(z_j).$$

PROOF. Use (4), (6), and the Residue Theorem. $\square$

III.F.18. EXAMPLE. $\oint_{|z|=2} \frac{10z^9 g(z)}{z^{10}-1} dz = 2\pi i \sum_{j=1}^{10} g(e^{2\pi i j/10}).$

These are amusing exercises, but the real power of these results first becomes evident from using them to compute real integrals:

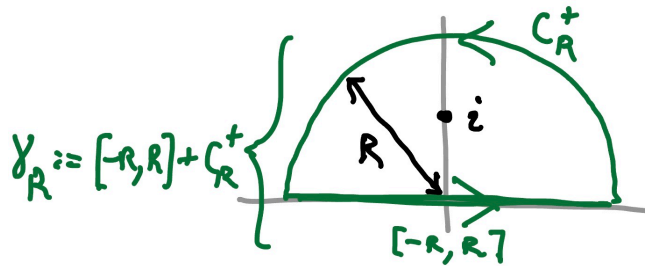
III.F.19. EXAMPLE. Set

$$I := \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2}{(x^2+1)^2} dx.$$

We will apply the residue theorem to the integral of

$$f(z) := \frac{z^2}{(z^2+1)^2} \in \text{Hol}(\mathbb{C} \setminus \{i, -i\})$$

over the closed path shown:



Notice that

$$\left| \int_{C_R^+} f \, dz \right| \leq \pi R \|f\|_{C_R^+} \leq \frac{\pi R^3}{(R^2-1)^2} = \frac{\pi R^3}{R^4 - 2R^2 + 1} \xrightarrow{R \rightarrow \infty} 0,$$

since for $|z| = R$ we have $|z^2+1| \geq |z|^2-1 = R^2-1$. So

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \left(\oint_{\gamma_R} f \, dz - \int_{C_R^+} f \, dz \right) = \lim_{R \rightarrow \infty} \oint_{\gamma_R} f \, dz \\ &= 2\pi i \text{Res}_i(f). \end{aligned}$$

To evaluate this residue we use method (2) above with $n = 2$: writing

$$f(z) = \frac{z^2/(z+\mathbf{i})^2}{(z-\mathbf{i})^2} = \frac{F(z)}{(z-\mathbf{i})^2},$$

we have

$$\begin{aligned} F'(z) = \frac{2iz}{(z+\mathbf{i})^3} &\implies F'(\mathbf{i}) = -\frac{\mathbf{i}}{4} \implies \text{Res}_{\mathbf{i}}(f) = \frac{F'(\mathbf{i})}{(2-1)!} = -\frac{\mathbf{i}}{4} \\ \implies I = 2\pi\mathbf{i} \left(-\frac{\mathbf{i}}{4}\right) &= \frac{\pi}{2}. \end{aligned}$$

III.F.20. DEFINITION. A closed curve γ **has an interior**¹³ if $W(\gamma, \alpha) = 0$ or 1 for every $\alpha \notin \gamma$, and $\text{Int}(\gamma) := \{\alpha \in \mathbb{C} \mid W(\gamma, \alpha) = 1\}$.

Let $f \in \text{Mer}(U)$ and $g \in \text{Hol}(U)$ for some open set containing both γ and $\text{Int}(\gamma)$. Assume no poles or zeroes of f lie on γ .

III.F.21. COROLLARY (of the Argument Principle). *If γ has an interior, then*

$$\frac{1}{2\pi\mathbf{i}} \oint_{\gamma} \frac{f'}{f} dz = \sum_{p \in \text{Int}(\gamma)} \text{ord}_p(f) = \left(\begin{array}{c} \# \text{ of zeroes of } f \\ \text{inside } \gamma \end{array} \right) - \left(\begin{array}{c} \# \text{ of poles of } f \\ \text{inside } \gamma \end{array} \right).$$

In the right-hand expression, the zeroes and poles are counted with multiplicity (a pole of order 2 counts as 2 poles).

III.F.22. COROLLARY (of the GAP). *If γ has an interior, then*

$$\frac{1}{2\pi\mathbf{i}} \oint_{\gamma} \frac{f'}{f} g dz = \sum_{p \in \text{Int}(\gamma)} \text{ord}_p(f) g(p).$$

Note that Cauchy's integral formula is a special case of Cor. III.F.22, with $f(z) = z - z_0$.

¹³This is a slightly more general condition than being simple.

§ Residues of differentials.

It sometimes makes more sense to write “ $d\log(f)$ ” instead of $\frac{f'(z)}{f(z)}dz$. Recall that this is *not* necessarily d of some function $\log f$.¹⁴ We have the obvious properties

- $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g} \implies d\log(fg) = d\log(f) + d\log(g)$.
- $d\log(\alpha f) = d\log(f)$
- $d\log(f/g) = d\log(f) - d\log(g)$.

This goes beyond a mnemonic for logarithmic differentiation rules: first, for the above Corollary of GAP, it expresses the fact that *you are recording how much $\log(f)$ (or $\arg(f)$) changes around γ* . But it also is because of the following

III.F.23. FACT. *Residues of functions are not invariant under local analytic isomorphism. Residues of differentials are.*

Here $\text{Res}_{z_0}(F(z)dz) := \text{Res}_{z_0}(F(z))$, so they appear to be exactly the same. But now substitute

$$z = z(w) = z_0 + b_1(w - w_0)(1 + h(w))$$

(with $b_1 \neq 0$ and $h(w_0) = 0$) and take Res_{w_0} :

- $F(z) \rightsquigarrow F(z(w))$ yields $\text{Res}_{w_0}(F(z(w)))$;
- $F(z)dz \rightsquigarrow F(z(w))z'(w)dw$, on the other hand, yields

$$\text{Res}_{w_0}(F(z(w))z'(w)dw) = \text{Res}_{w_0}(F(z(w))z'(w)).$$

Obviously, these *can't* be the same in general.

III.F.24. EXAMPLE. The simplest function with a nontrivial residue is $F(z) = a_{-1}(z - z_0)^{-1}$. Let $z(w) = z_0 + b_1(w - w_0)$, $z'(w) = b_1$, be our change of variable. Then $F(z(w)) = a_{-1}b_1^{-1}(w - w_0)^{-1}$, but $F(z(w))z'(w) = a_{-1}b_1^{-1}(w - w_0)^{-1}b_1 = a_{-1}(w - w_0)^{-1}$. It's the second expression whose residue remains equal to a_{-1} .

¹⁴This is locally true, but need not be globally true on U .

To see this more generally and conceptually, write

$$\begin{aligned}\operatorname{Res}_{w_0}(F(z(w))z'(w)[dw]) &= \frac{1}{2\pi i} \oint_{\gamma} F(z(w))z'(w)dw \\ &= \frac{1}{2\pi i} \oint_{z \circ \gamma} F(z)dz = \operatorname{Res}_{z_0}(F(z)[dz])\end{aligned}$$

where



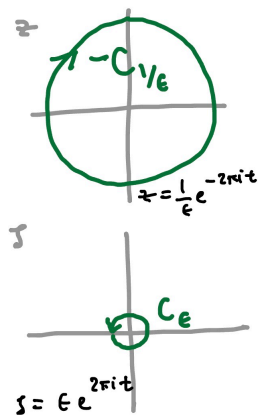
So the differential works because it transforms like the differential in the integral under change of variable.

Residue at ∞ . Here we transform the *differential* into the local coordinate at ∞ , $\zeta = \frac{1}{z}$ ($\leftrightarrow z = \frac{1}{\zeta}$):

$$f(z)dz = f\left(\frac{1}{\zeta}\right)d\left(\frac{1}{\zeta}\right) = -f\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2},$$

where f (say) is meromorphic on $\mathbb{C}$. Now assume f has finitely many poles in $\mathbb{C}$, so that for some $\epsilon > 0$, the only pole enclosed by C_ϵ (in ζ) is at $\zeta = 0$ (if there even is one). Then

$$\begin{aligned}\operatorname{Res}_\infty(f(z)dz) &:= \operatorname{Res}_0\left(-f\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2}\right) \\ &= \operatorname{Res}_0\left(\frac{-f(1/\zeta)}{\zeta^2}\right) \\ &= \frac{1}{2\pi i} \int_{C_\epsilon} -f\left(\frac{1}{\zeta}\right)\frac{d\zeta}{\zeta^2} \\ &= \frac{1}{2\pi i} \int_{-C_{1/\epsilon}} f(z)dz \\ &= -\frac{1}{2\pi i} \int_{C_R} f(z)dz\end{aligned}$$



where $R = 1/\epsilon$. By construction C_R encloses all poles of f in $\mathbb{C}$. But then

$$\begin{aligned} \sum_{p \in \hat{\mathbb{C}}} \text{Res}_p(f(z)dz) &= \sum_{p \in \mathbb{C}} \text{Res}_p(f(z)dz) + \text{Res}_\infty(f(z)dz) \\ &= \frac{1}{2\pi i} \int_{C_R} f(z)dz + \left(-\frac{1}{2\pi i} \int_{C_R} f(z)dz \right) \\ &= 0 \end{aligned}$$

and we conclude the

III.F.25. THEOREM. *The sum of the residues of a differential on $\hat{\mathbb{C}}$ is 0.*