III.G. Rouché's theorem

§ Counting zeroes.

Recall that a closed path $\gamma \subset \mathbb{C}$ is said to "have an interior" if $W(\gamma, \alpha) = 0$ or 1 $(\forall \alpha \in \mathbb{C} \setminus \gamma)$; and then

$$\mathrm{Int}(\gamma) := \{\alpha \in \mathbb{C} \setminus \gamma \mid W(\gamma, \alpha) = 1\}.$$

For such a γ, let U be an open set containing $\overline{\mathrm{Int}(\gamma)} = \mathrm{Int}(\gamma) \cup \gamma$, and $f \in \mathscr{Mer}(U)$ have no zeroes or poles on γ itself. Then we had (a special case of) the argument principle

$$\frac{1}{2\pi\mathbf{i}} \int_\gamma \frac{f'}{f}\, dz = \sum_{p \in \mathrm{Int}(\gamma)} \mathrm{ord}_p(f)$$

and of its generalization (with $g \in \mathscr{Hol}(U)$)

$$\frac{1}{2\pi\mathbf{i}} \int_\gamma g\frac{f'}{f}\, dz = \sum_{p \in \mathrm{Int}(\gamma)} g(p)\, \mathrm{ord}_p(f),$$

which we abbreviate GAP. Here $\sum \mathrm{ord}_p(f)$ has the standard interpretation as $(\#\text{ of zeroes w/mult.}) - (\#\text{ of poles w/mult.})$ enclosed by γ.

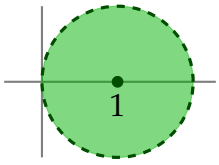
The following application of the argument principle is very useful:

III.G.1. ROUCHÉ'S THEOREM. *If γ has an interior, $\overline{\mathrm{Int}(\gamma)} \subset U$ $(\implies \gamma \equiv_{\mathrm{hom}} 0$ on $U)$, and $f, g \in \mathscr{Hol}(U)$ satisfy*

$$|f - g| < |f| \ \text{ on } \ \gamma, \tag{III.G.2}$$

then f and g have the same number of zeroes (counted with multiplicity) in $\mathrm{Int}(\gamma)$.

PROOF. From (III.G.2) we see that on γ, f and g are nowhere zero and $|\frac{g}{f} - 1| < 1$. That is, the values of $\frac{g}{f}$ on γ lie in $D(1, 1)$.



It follows that $W(\frac{g}{f} \circ \gamma, 0) = 0$, hence

$$0 = \frac{1}{2\pi \mathbf{i}} \int_{\frac{f}{g} \circ \gamma} \operatorname{dlog}(z) = \frac{1}{2\pi \mathbf{i}} \int_{\gamma} \operatorname{dlog}\left(\frac{f}{g}\right)$$

$$= \frac{1}{2\pi \mathbf{i}} \int_{\gamma} \operatorname{dlog}(f) - \frac{1}{2\pi \mathbf{i}} \int_{\gamma} \operatorname{dlog}(g)$$

$=$ (# zeroes f counted w/mult.) $-$ (# zeroes g counted w/mult.) $\square$

III.G.3. EXAMPLE. Consider a polynomial with one BIG coefficient and the others small:

$$P(z) = a_d z^d + \cdots + a_1 z + a_0 + A z^m, \quad |A| > \sum |a_i|,$$

where $0 \leq m \leq d$. How many zeroes does $P(z)$ have inside the unit circle?

Set $g(z) := P(z)$, $f(z) := Az^m$, so that $g(z) - f(z) = \sum a_j z^j$. On C_1, we therefore have

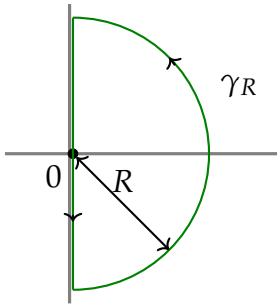
$$|g - f| \leq \sum |a_j| \cancel{|z|^j}^{1} < |A| = |f|,$$

and Rouché applies, whence f and g have the same number of zeroes with multiplicity in D_1. Since f has one zero of multiplicity m at $z = 0$, we conclude that

$P(z)$ has m zeroes (counted with multiplicity) inside the unit disc.

III.G.4. EXAMPLE. How many solutions does $z^{2n+1} + e^{-z} = \lambda$ have in the right half-plane, for $\lambda \in \mathbb{R}_{>1}$?

Set $f(z) = z^{2n+1} - \lambda$, $g(z) = z^{2n+1} + e^{-z} - \lambda$ ($\implies f - g = -e^{-z}$) and consider their behavior on γ_R:



we have $|f-g| = |e^{-z}| = \frac{1}{e^x} \leq 1$ and $(1 <)\, \lambda \leq |f|$ there for $R \gg 0$. To see the latter, note that

- $z \in \mathbf{i}\mathbb{R} \implies z^{2n+1} \in \mathbf{i}\mathbb{R} \implies |z^{2n+1} - \lambda| = \sqrt{\lambda^2 + |z^{2n+1}|^2} \geq \lambda$
- $z \in$ semicircle $\implies$ we can take R sufficiently large that $|z|^{2n+1}$ exceeds 2λ.

So taking $R \to \infty$, Rouché tells us that f and g have the same number of zeroes (counted with multiplicity) in the right half-plane.

Now $f(z) = 0 \iff z = \sqrt[2n+1]{\lambda} \cdot \zeta_{2n+1}^j$, and n [resp. $n+1$] of these (simple) zeroes lie in the right half-plane for n odd [resp. even]. So g has n [resp. $n+1$] zeroes there too.

§ Inverse functions.

Rouché's theorem provides a neat proof of the inverse mapping theorem. Let U be an open set and $z_0 \in U$.

III.G.5. THEOREM. *If $F \in \mathscr{Hol}(U)$ and $F'(z_0) \neq 0$, then F is a local analytic isomorphism at z_0.*

PROOF. We proceed in the following steps:

- We may assume $z_0 = 0$. Write $F(z) = z + z^2 h(z)$, with $h \in \mathscr{Hol}(U)$ bounded on some closed disk: $|h(z)| \leq K$ for $z \in \overline{D}_R$.
- Take $\epsilon < \frac{1}{2K}$, let $|w| < \frac{\epsilon}{2}$; and set

$$g(z) := F(z) - w\,, \quad f(z) := z - w.$$

- For $z \in C_\epsilon$, $|f| = |z-w| \geq |z| - |w| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}$, and so $|g-f| = |F(z) - z| \leq K|z|^2 = K\epsilon^2 < \frac{\epsilon}{2} < |f|$.
- By Rouché, g and f have the same number of zeroes (w/mult.) in $D_\epsilon = \mathrm{Int}(C_\epsilon)$; f has one (simple) zero at $z = w$, so g has exactly one zero (also simple).

Conclusion: For $|w| < \frac{\epsilon}{2}$ (i.e., sufficiently small), there exists a unique $z \in D_\epsilon$ such that $F(z) = w$, giving an inverse

$$F^{-1} \colon D_{\epsilon/2} \to D_\epsilon$$
$$w \mapsto F^{-1}(w) = z.$$

Is this F^{-1} holomorphic? We need this to conclude that F is a local analytic isomorphism. The GAP (now with $f(z) = F(z) - w$ and $g(z) = z$) tells us that

$$\text{(III.G.6)}\quad \frac{1}{2\pi\mathbf{i}}\int_{C_\epsilon}\frac{F'(z)}{F(z)-w}z\,dz = \sum_{p\in D_\epsilon}\mathrm{ord}_p(F(z)-w)\cdot z(p) = F^{-1}(w),$$

as there is a unique $p \in D_\epsilon$ with $F(p) - w = 0$, namely $p = F^{-1}(w)$. Since the integral representation on LHS(III.G.6) is automatically holomorphic, we are done. □

Similarly, the GAP gives us the analyticity of symmetric polynomials in the values of "multivalued inverses". Assume that

$$f(z) = w \quad \textit{has } n \textit{ solutions } \{z_j(w)\} \textit{ in } D(z_0,\epsilon) \textit{ for } |w - w_0| < \delta,$$

which may be multivalued in w (e.g. with monodromy about w_0). By the GAP with $g(z) = z^m$, the m^{th} **Newton symmetric polynomial**

$$p_m(\{z_j(w)\}) := \sum_{j=1}^{n}(z_j(w))^m = \frac{1}{2\pi\mathbf{i}}\int_{\partial D(z_0,\epsilon)} z^m\frac{f'(z)}{f(z)-w}\,dz$$

in them is a single-valued function in $\mathscr{Hol}(D(w_0,\delta))$. Formally writing (for $(z,w) \in D(z_0,\epsilon)\times D(w_0,\delta)$)

$$\begin{aligned} f(z) - w &= H(z,w)\cdot\textstyle\prod_{j=1}^{n}(z - z_j(w)) \\ &= H(z,w)\cdot\{z^n - e_1(\{z_j(w)\})z^{n-1} + e_2(\{z_j(w)\})z^{n-2} \\ &\qquad\qquad - \cdots + (-1)^n e_n(\{z_j(w)\})\}, \end{aligned} \quad \text{(III.G.7)}$$

with $H(z,w)$ holomorphic and nowhere vanishing, defines the **elementary symmetric polynomials** in the $\{z_j(w)\}$, with e_1 their sum and e_n their product. A well-known fact from algebra is that the $\{e_k\}$ are linear combinations of the $\{p_m\}$. Hence they are also single-valued and holomorphic, and (writing $E_k(w) := e_k(\{z_j(w)\})$) we may replace "$f(z) = w$" by

$$0 = z^n - E_1(w)z^{n-1} + E_2(w)z^{n-2} + \cdots + (-1)^n E_n(w).$$

The factorization (III.G.7) is called a **Weierstrass factorization**.

§ Irreducible algebraic curves are connected.

In the remainder of this chapter we take an algebro-geometric detour, turning to two important applications of Rouché and Riemann to complex algebraic curves. Below *I will use x and y to denote two complex variables* rather than the real and imaginary parts of a single complex variable.

III.G.8. DEFINITION. An **affine algebraic curve** $C \subset \mathbb{C}^2$ is the solution set of a polynomial equation of the form[15]

$$0 = f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \tag{III.G.9}$$

where the $a_i(x)$ are polynomials (in x) of degree $\leq i$.

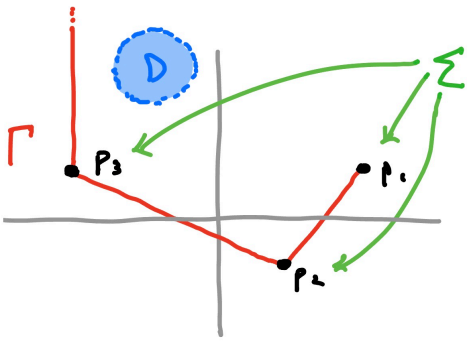
Projecting

$$\pi : C \to \mathbb{C}$$
$$(x,y) \mapsto x,$$

we find that on the complement of a finite point set $\Sigma \subset \mathbb{C}$, the preimage consists of exactly n points:

$$\pi^{-1}(x) = \{y_1(x), \ldots, y_n(x)\}. \tag{III.G.10}$$

More precisely, one writes (III.G.10) on a small disk $D \subset \mathbb{C} \setminus \Sigma$ and analytically continues the resulting $\{y_j(x)\}$ to the simply connected complement of a set Γ (as shown): even if continued *through* $\Gamma \setminus \Sigma$, these analytically continued $\{y_i(x)\}$ still satisfy



$$f(x, y_i(x)) = 0.$$

But *they may permute!* (Think back to Riemann surfaces.) The $\{y_i(x)\}$ are called **branches**, and they are exchanged as we travel around loops in $\mathbb{C} \setminus \Sigma$ and back to D. We can think of this as a group homomorphism from $\pi_1(\mathbb{C} \setminus \Sigma)$ to the n^{th} symmetric group $\mathfrak{S}_n$ which is called the **monodromy** action.

[15] More precisely, the equation can always be put in this form by a change of variable of the form $(x_0, y_0) \mapsto (x, y) = (x_0 - \lambda y_0, y_0)$ and rescaling.

III.G.11. PROPOSITION. *Let $E \subsetneq \{1,\dots,n\}$ be a proper subset which is closed under this monodromy action (with respect to continuation over* all *of $\mathbb{C}\setminus\Sigma$). Relabeling, we may assume that it takes the form $\{1,\dots,m\}$ (with $m<n$). Then $\prod_{\ell=1}^{m}(y-y_\ell(x))$ is a polynomial in x and y, and so* (III.G.9) *factors nontrivially in $\mathbb{C}[x,y]$.*

III.G.12. THEOREM. *If C is irreducible (i.e., f is irreducible in $\mathbb{C}[x,y]$), then $C\setminus\pi^{-1}(\Sigma)$ is connected (and therefore so is C).*

PROOF THAT PROP. III.G.11 $\implies$ THM. III.G.12. Suppose f is irreducible. Then by the Proposition, the only subset of $\{1,\dots,n\}$ closed under the monodromy action $\pi_1(\mathbb{C}\setminus\Sigma)\to\mathfrak{S}_n$ is $\{1,\dots,n\}$ itself.[16] So the complete set of branches $\{y_i(x)\}$ is acted on transitively by monodromy about Σ, and one can therefore draw a path on $C\setminus\pi^{-1}(\Sigma)$ connecting any two points. □

PROOF OF PROP. III.G.11. The product $\prod_{\ell=1}^{m}(y-y_\ell(x))$ is clearly well-defined for $x\in\mathbb{C}\setminus\Sigma$, since continuation through $\Gamma\setminus\Sigma$ (i.e., monodromy about Σ) merely permutes its factors. Writing

$$\prod_{\ell=1}^{m}(y-y_\ell(x)) = \sum_{j=0}^{m} e_{m-j}(\{y_\ell(x)\}_{\ell=1}^{m})\cdot y^j =: \sum_{j=0}^{m} E_{m-j}(x)y^j,$$

the $E_k(x)$ belong to $\mathscr{Hol}(\mathbb{C}\setminus\Sigma)$ as in the last section, and so for each fixed y the product is also in $\mathscr{Hol}(\mathbb{C}\setminus\Sigma)$. But we need to show that the $E_k(x)$ are *polynomials*. This is where Rouché comes in.

Let $\alpha\in\Sigma$ and $\mathcal{N}_\alpha$ be a small disc about α. We have

$$x\in\mathcal{N}_\alpha \implies |a_j(x)|\le M \quad(\forall j).$$

Fix $x_0\in\mathcal{N}_\alpha^*$, put $a_j := a_j(x_0)$; then we shall apply Rouché to

$$\mathfrak{F}(y)=y^n, \qquad \mathfrak{G}(y)=y^n+a_1y^{n-1}+\cdots+a_n,$$

$$\text{and}\quad \gamma=\{|y|=M+1\}\subset\mathbb{C}.$$

[16] Here $\mathfrak{S}_n$ denotes the n^{th} symmetric group, which consists of permutations of n elements.

They satisfy the hypotheses since (on γ)

$$\begin{aligned}|\mathfrak{F}-\mathfrak{G}| = |a_1 y^{n-1} + \cdots + a_n| &\leq M((M+1)^{n-1} + \cdots + 1) \\ &= (M+1)^n - 1 \\ &< (M+1)^n = |\mathfrak{F}|.\end{aligned}$$

Thus $\mathfrak{F}$ and $\mathfrak{G}$ have the same number of zeroes inside γ: this is n for $\mathfrak{F}$, and thus also for $\mathfrak{G}$. It follows that $|y_j(x_0)| < M+1$ for all $j = 1, \ldots, n$ and $x_0 \in \mathcal{N}_\alpha^*$, whence the $E_k(x)$ are bounded (as well as single-valued) on $\mathcal{N}_\alpha^*$. By Riemann, their "singularity" at α is removable, and so $E_k \in \mathscr{Hol}(\mathcal{N}_\alpha)$ for each k. Since $\alpha \in \Sigma$ was arbitrary, and the union of $\mathbb{C} \setminus \Sigma$ and the $\mathcal{N}_\alpha$ is all of $\mathbb{C}$, we conclude that the E_k are entire.

To show that they are polynomials, we must analyze their behavior *about* ∞ to rule out an essential singularity there. Changing coordinates to $\tilde{x} = \frac{1}{x}$, $\tilde{y} = \frac{y}{x}$, (III.G.9) becomes

$$0 = \tilde{x}^n f\left(\frac{1}{\tilde{x}}, \frac{\tilde{y}}{\tilde{x}}\right) = \tilde{y}^n + \left[\tilde{x} a_1\left(\frac{1}{\tilde{x}}\right)\right]\tilde{y}^{n-1} + \cdots + \left[\tilde{x}^n a_n\left(\frac{1}{\tilde{x}}\right)\right]$$

where the bracketed expressions are polynomials in $\tilde{x}$. The roots of this equation are

$$\tilde{y}_i(\tilde{x}) = \tilde{x} \cdot y_i\left(\frac{1}{\tilde{x}}\right).$$

From this formula, clearly the $\{\tilde{y}_i\}_{i=1}^m$ are permuted amongst themselves by monodromy. So the above argument applies to

$$e_k(\{\tilde{y}_l(\tilde{x}), \ldots, \tilde{y}_m(\tilde{x})\})$$

on a neighborhood $\mathcal{N}_\infty$. That is,

$$\mathscr{Hol}(\mathcal{N}_\infty) \ni e_k(\{\tilde{y}_\ell(\tilde{x})\}_{\ell=1}^m) = \tilde{x}^k e_k\left(\left\{y_\ell\left(\tfrac{1}{\tilde{x}}\right)\right\}_{\ell=1}^m\right) = \frac{E_k(x)}{x^k}$$

and so $E_k(x)$ has at worst a pole of order k at ∞. By Corollary III.E.19(a), it is a polynomial (of degree at most k in x). □

§ Meromorphic functions on algebraic curves are rational.

In Theorem III.E.17, we saw that meromorphic functions on the Riemann sphere are rational, i.e. quotients of polynomials. The idea is that the compactness of $\hat{\mathbb{C}}$ together with the strong constraint imposed by meromorphicity does the job in much the same way that boundedness and holomorphicity force constancy in Liouville's theorem. A complete, nonsingular algebraic curve over $\mathbb{C}$ is the same thing as a compact Riemann surface, and the result on $\hat{\mathbb{C}}$ generalizes to this case, as we now shall demonstrate.

Taking C now to be irreducible, and starting with the same projection

$$\pi\colon C \to \mathbb{C},$$

we have inclusions of fields

$$\pi^*\mathbb{C}(x) \subseteq \mathbb{C}(C) \subseteq \mathscr{Mer}(\overline{C}). \tag{III.G.13}$$

Here $\mathbb{C}(x)$ is the field of rational functions in x, and π^* means their pullback to C (think $\frac{P}{Q} \circ \pi$); while $\mathbb{C}(C)$ denotes the functions obtained by restricting rational functions in x and y to C. Finally, $\overline{C}$ is the closure of C in the complex projective plane[17] $\mathbb{P}^2$; this is the compact Riemann surface on which we want to consider meromorphic functions.

III.G.14. CLAIM. *The degrees of field extensions in* (III.G.13) *satsify:*

(a) $[\mathbb{C}(C) : \pi^*\mathbb{C}(x)] \geq n$
(b) $[\mathscr{Mer}(\overline{C}) : \pi^*\mathbb{C}(x)] \leq n$

The next result follows immediately from (III.G.13), the Claim, and the tower law for field extensions (that degrees of successive extensions multiply):

III.G.15. THEOREM. $\mathscr{Mer}(\overline{C}) = \mathbb{C}(C)$

[17] This is the compactification of $\mathbb{C}^2$ by adding a "line at ∞". We can assume that this closure is smooth.

PROOF OF CLAIM. (a) is trivial: by irreducibility of C,

$$0 = f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$$

is the minimal polynomial of y over $\pi^*\mathbb{C}(x)$. Since $\mathbb{C}(C)$ contains an element of degree n over the subfield, the extension has at least that degree.

More interesting is the proof of (b). Let $\varphi \in \mathscr{Mer}(\overline{C})^*$, and denote by $\mathcal{P}$ the image under π of its polar set and $\tilde{\Sigma} = \Sigma \cup \mathcal{P}$. Write

$$e_i^\varphi(x) := e_i(\{\varphi(x,y_1(x)), \ldots, \varphi(x,y_n(x))\}) \in \mathscr{Hol}(\mathbb{C} \setminus \tilde{\Sigma}).$$

The same argument as above involving Riemann and Rouché implies that these e_i^φ extend to holomorphic functions on $\mathbb{C} \setminus \mathcal{P}$.

About a point $\beta \in \mathcal{P}$, we can look at e_i^φ times a sufficiently high power of $(x-\beta)$; this cancels out the pole of φ at $(\beta, y_j(\beta))$ and allows the same argument to be applied, with the consequence that e_i^φ has meromorphic extension to $\mathbb{C}$. Again, analyzing about ∞ yields $e_i^\varphi \in \mathscr{Mer}(\hat{\mathbb{C}}) \cong \mathbb{C}(x)$.

Now for any $x \in \mathbb{C} \setminus \tilde{\Sigma}$ and $j \in \{1, \ldots, n\}$,

$$0 = \prod_{i=1}^n (\varphi(x,y_j(x)) - \varphi(x,y_i(x))) = \sum_{k=0}^n \varphi(x,y_j(x))^{n-k} \cdot (-1)^k e_k^\varphi(x).$$

So the meromorphic function φ itself satisfies

$$0 = \sum_{k=0}^n (-1)^k (\pi^* e_k^\varphi) \cdot \varphi^{n-k},$$

showing that any element of $\mathscr{Mer}(\overline{C})^*$ has degree $\leq n$ over $\pi^*\mathbb{C}(x)$. If the extension $\mathscr{Mer}(\overline{C})/\pi^*\mathbb{C}(x)$ had degree $d > n$, then (by an important result in algebra) there would be a "primitive element" of degree d. We have just seen that there isn't. □

Exercises

(1) Let $P_n(z) = \sum_{k=0}^n \frac{z^k}{k!}$. Given R, prove that P_n has no zeroes in the disk of radius R for all n sufficiently large.

(2) How many zeroes does $f(z) = z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1$ have in the annulus $1 < |z| < 2$? [Hint: Rouché, twice.]