III.H. Computing real integrals

The most elementary motivation for residue calculus is to apply path integrals and residues in $\mathbb{C}$ to the computation of real integrals.

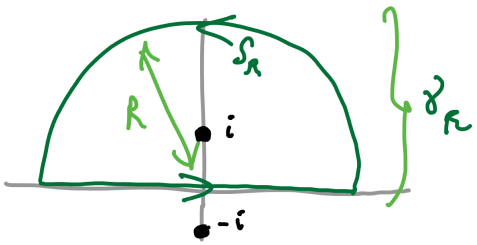
§ Rational integrals.

We saw the example that follows in Chapter III.F, but I'd like to review it to illustrate a more general principle.

III.H.1. EXAMPLE. To compute $I = \int_0^\infty \frac{x^2}{(1+x^2)^2}dx$, write

$$2I = \int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^2}dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{x^2}{(1+x^2)^2}dx,$$

set $f(z) := \frac{z^2}{(z^2+1)^2} \in \mathscr{H}\!ol(\mathbb{C}\setminus\{\mathbf{i},-\mathbf{i}\})$, and consider the path



On S_R we have

$$|z^2+1| \geq |z|^2 - 1 = R^2 - 1 \quad\Longrightarrow\quad \frac{|z|^2}{|z^2+1|^2} \leq \frac{R^2}{(R^2-1)^2}$$

whence

$$\left|\int_{S_R} f dz\right| \leq L(S_R)\|f\|_{S_R} \leq \pi R \cdot \frac{R^2}{(R^2-1)^2} = \frac{\pi R^3}{R^4 - 2R^2 + 1} \to 0$$

as $R \to \infty$. The residue theorem now gives

$$2I = \lim_{R\to\infty}\left(\int_{\gamma_R} f\,dz - \int_{S_R} f\,dz\right) = \lim_{R\to\infty}\int_{\gamma_R} f\,dz = 2\pi\mathbf{i}\mathrm{Res}_{\mathbf{i}}(f),$$

and writing

$$f(z) = \frac{z^2/(z+\mathbf{i})^2}{(z-\mathbf{i})^2} =: \frac{g(z)}{(z-\mathbf{i})^2},$$

we compute

$$g'(z) = \frac{2z(z+\mathbf{i})^{\cancel{2}} - 2z^2\cancel{(z+\mathbf{i})}}{(z+\mathbf{i})^{\cancel{4}\,3}} = \frac{2\mathbf{i}z}{(z+\mathbf{i})^3} \quad \Longrightarrow \quad g'(\mathbf{i}) = \frac{-\mathbf{i}}{4}.$$

Conclude that

$$I = \pi\mathbf{i}\mathrm{Res}_{\mathbf{i}}(f) = \pi\mathbf{i} \cdot \frac{g'(\mathbf{i})}{(2-1)!} = \pi\mathbf{i} \cdot \left(\frac{-\mathbf{i}}{4}\right) = \frac{\pi}{4},$$

which is real and positive (as expected).

Exactly the same reasoning leads to the following result; note that the $\int_{-\infty}^{\infty}$ in question converges by the assumption on degrees.

III.H.2. THEOREM. *Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function with*

$$\deg(Q) \geq 2 + \deg(P),$$

and all roots of Q non-real. Then if $z_j = x_j + \mathbf{i}y_j$ are the roots of $Q(z)$, we have

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi\mathbf{i} \sum_{j:y_j>0} \mathrm{Res}_{z_j}(f).$$

III.H.3. REMARK. More generally, the result holds provided $f(x)$ is the restriction to $\mathbb{R}$ of $f(z) \in \mathscr{Mer}(\overline{\mathfrak{H}})$ with only finitely many poles (in $\mathfrak{H}$) and none on $\mathbb{R}$, and if there exist $a > 1$ and $B, K > 0$ such that

$$|f(z)| \leq \frac{B}{|z|^a} \text{ for } |z| \geq K \text{ (and } z \in \overline{\mathfrak{H}}).$$

PROOF. We only need to show that the integral over the big semi-circle vanishes as $R \to \infty$:

$$\lim_{R\to\infty} \left| \int_{S_R} f(z)dz \right| \leq \lim_{R\to\infty} \pi R \cdot \frac{B}{R^a} = \lim_{R\to\infty} \frac{\pi B}{R^{a-1}} = 0.$$

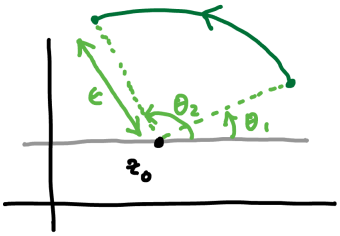
This covers the Remark, of which the Theorem is a special case with $a = \deg(Q) - \deg(P)$. □

§ How to deal with a pole on $\mathbb{R}$.

Consider the integral

$$\mathrm{I} = \int_0^\infty \frac{\sin(x)}{x} dx.$$

Of course, $\frac{\sin(z)}{z}$ doesn't have a pole, but bear with me.



III.H.4. LEMMA. *If $f \in \mathscr{Hol}(\overline{D}^*(z_0, r))$ has a* simple *pole at z_0, and $A_{\theta_1}^{\theta_2}(z_0, \epsilon)$ is the path (arc) displayed, then*[18]

$$\lim_{\epsilon \to 0} \int_{A_{\theta_1}^{\theta_2}(z_0,\epsilon)} f(z)dz = \mathbf{i}(\theta_2 - \theta_1)\mathrm{Res}_{z_0}(f).$$

PROOF. We write $D = D(z_0, r)$ and $f(z) = \frac{\mathrm{Res}_{z_0}(f)}{z - z_0} + h(z)$, with $h \in \mathscr{Hol}(\overline{D})$ and $\|h\|_{\overline{D}} \leq B$. Then parametrizing by $\gamma(\theta) = \epsilon e^{\mathbf{i}\theta} + z_0$,

$$\begin{aligned}
\int_{A_{\theta_1}^{\theta_2}(z_0,\epsilon)} f(z)dz &= \int_{\theta_1}^{\theta_2} f(\gamma(\theta))\gamma'(\theta)d\theta \\
&= \int_{\theta_1}^{\theta_2} \frac{\mathrm{Res}_{z_0}(f)}{\cancel{\epsilon e^{\mathbf{i}\theta}}} \mathbf{i}\cancel{\epsilon} e^{\cancel{\mathbf{i}\theta}} d\theta + \int_{\theta_1}^{\theta_2} h(\gamma(\theta))\mathbf{i}\epsilon e^{\mathbf{i}\theta} d\theta \\
&= \mathbf{i}(\theta_2 - \theta_1)\mathrm{Res}_{z_0}(f) + \mathbf{i}\epsilon \int_{\theta_1}^{\theta_2} e^{\mathbf{i}\theta} h(\gamma(\theta))d\theta.
\end{aligned}$$

The modulus of the last term is bounded by $\epsilon B(\theta_2 - \theta_1)$, which goes to 0 with ϵ. □

As $\frac{\sin(x)}{x}$ is even, we have

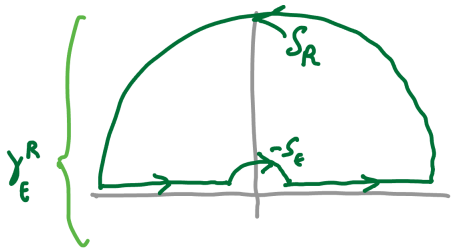
$$2\mathrm{I} = \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \lim_{\epsilon \to 0} \left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{\sin(x)}{x} dx \right];$$

[18] Here we take the difference of angles in radians.

and as $(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty})\frac{\cos(x)}{x}dx = 0$ and $\frac{\sin(x)}{x} = -\frac{\mathbf{i}e^{\mathbf{i}x}}{x} + \frac{\mathbf{i}\cos(x)}{x}$, this

$$= \lim_{\epsilon\to 0}\left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{-\mathbf{i}e^{\mathbf{i}x}}{x}dx\right] =: -\mathbf{i}\cdot \mathrm{PV}\left(\int_{-\infty}^{\infty}\frac{e^{\mathbf{i}x}}{x}dx\right).$$

Here "PV" stands for the **principal value** of an integral, which may exist even (as in this case) when the integral itself is not well-defined. The culprit here is the behavior of $\frac{\cos(x)}{x}$ at the origin. That's the pole we now have to deal with.



Now $\frac{e^{\mathbf{i}z}}{z}$ has no poles in $\mathfrak{H}$, so for all $0 < \epsilon < R < \infty$ we have

$$\begin{aligned} 0 = \int_\gamma \frac{e^{\mathbf{i}z}}{z}dz &= \lim_{\epsilon\to 0}\lim_{R\to\infty}\left\{\left(\int_{S_R} + \int_{-S_\epsilon} + \int_{-R}^{-\epsilon} + \int_{\epsilon}^{R}\right)\frac{e^{\mathbf{i}z}}{z}dz\right\} \\ &= \mathrm{PV}\left[\int_{-\infty}^{\infty}\frac{e^{\mathbf{i}x}}{x}dx\right] + \lim_{R\to\infty}\int_{S_R}\frac{e^{\mathbf{i}z}}{z}dz + \lim_{\epsilon\to 0}\int_{-S_\epsilon}\frac{e^{\mathbf{i}z}}{z}dz \\ &=: 2\mathbf{i}\cdot \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

By the Lemma, we have $\mathrm{III} = \mathbf{i}(0-\pi)\,\mathrm{Res}_0\left(\frac{e^{\mathbf{i}z}}{z}\right) = -\pi\mathbf{i}$. For II, use $\gamma(\theta) = Re^{\mathbf{i}\theta} = R(\cos(\theta) + \mathbf{i}\sin(\theta))$ and $\gamma'(\theta) = R\mathbf{i}e^{\mathbf{i}\theta} = \mathbf{i}\gamma(\theta)$ to bound

$$\begin{aligned} \left|\int_{S_R}\frac{e^{\mathbf{i}z}}{z}dz\right| &= \left|\int_0^\pi \frac{e^{\mathbf{i}R\cos\theta}e^{-R\sin\theta}}{\cancel{\gamma(\theta)}}\mathbf{i}\cancel{\gamma(\theta)}d\theta\right| \\ &\le \int_0^\pi e^{-R\sin\theta}d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}d\theta \end{aligned}$$

Using $\sin(\theta) \ge \frac{2}{\pi}\theta$ on $[0, \frac{\pi}{2}]$, this is

$$\le 2\int_0^{\pi/2} e^{-2R\theta/\pi}d\theta = 2\left[-\frac{\pi}{2R}e^{-2R\theta/\pi}\right]_0^{\pi/2} = \frac{\pi}{R}(1-e^{-R}) \xrightarrow[R\to\infty]{} 0.$$

Hence $\mathrm{II} = 0$, and $0 = 2\mathbf{i}\cdot\mathrm{I} + \mathrm{III}$ becomes

$$\int_0^\infty \frac{\sin(x)}{x}dx = \mathrm{I} = \frac{\mathbf{i}}{2}\mathrm{III} = \frac{\mathbf{i}}{2}(-\pi\mathbf{i}) = \frac{\pi}{2}.$$

§ Trigonometric integrals.

Let $R(x,y)$ be a rational function in x and y, and consider

$$\mathscr{I} = \int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta.$$

We can make the substitution

$$\begin{cases} z = e^{\mathbf{i}\theta} \\ dz = \mathbf{i}e^{\mathbf{i}\theta} = \mathbf{i}z\,d\theta \\ -\mathbf{i}\frac{dz}{z} = d\theta \end{cases} \longleftrightarrow \begin{cases} \cos(\theta) = \frac{e^{\mathbf{i}\theta}+e^{-\mathbf{i}\theta}}{2} = \frac{1}{2}(z+z^{-1}) \\ \sin(\theta) = \frac{e^{\mathbf{i}\theta}-e^{-\mathbf{i}\theta}}{2\mathbf{i}} = \frac{1}{2\mathbf{i}}(z-z^{-1}) \end{cases}$$

to rewrite

$$\begin{aligned} \mathscr{I} &= -\mathbf{i}\oint_{C_1} R\left(\frac{1}{2}\left(z+z^{-1}\right), \frac{1}{2\mathbf{i}}\left(z-z^{-1}\right)\right)\frac{dz}{z} =: -\mathbf{i}\oint_{C_1} f(z)\,dz \\ &= -\mathbf{i}(2\pi\mathbf{i}\sum_{p\in D_1}\mathrm{Res}_p(f(z))) = 2\pi\sum_{p\in D_1}\mathrm{Res}_p(f(z)) \end{aligned}$$

using the residue theorem. (Of course, one should assume that the original integral was not improper.)

III.H.5. EXAMPLE. We shall compute

$$I = \int_0^{\pi} \frac{d\theta}{a+\cos\theta}$$

for $a \in \mathbb{R}_{>1}$, by applying the above to

$$\mathscr{I} = 2I = \int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \quad \text{with } R(x,y) = \frac{1}{a+x}.$$

So

$$f(z) = \frac{1}{z}\cdot\frac{1}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)} = \frac{1}{az+\frac{1}{2}z^2+\frac{1}{2}} = \frac{2}{z^2+2az+1}$$

has (simple) poles at

$$z_\pm = \frac{-2a\pm\sqrt{4a^2-4}}{2} = -a\pm\sqrt{a^2-1}.$$

Clearly

$$|z_-| = a+\sqrt{a^2-1} > 1,$$

while

$$\begin{aligned} 2 < 2a &\implies a^2 - 2a + 1 < a^-1 \\ &\implies a - 1 < \sqrt{a^2 - 1} \\ &\implies |z_+| = a - \sqrt{a^2 - 1} < 1 \\ &\implies z_+ \in D_1\,, \quad z_- \in D_1^c. \end{aligned}$$

Now

$$\begin{aligned} \mathrm{Res}_{z_+}(f) &= \mathrm{Res}_{z_+}\left(\frac{2/(z - z_-)}{z - z_+}\right) = \frac{2}{z_+ - z_-} = \frac{2}{2\sqrt{a^2-1}} \\ &= \frac{1}{\sqrt{a^2-1}}, \end{aligned}$$

hence $\mathscr{I} = 2\pi \mathrm{Res}_{z_+}(f) = \frac{2\pi}{\sqrt{a^2-1}}$

$$\implies \quad I = \frac{\mathscr{I}}{2} = \frac{\pi}{\sqrt{a^2-1}}.$$

* * *

In each of the exercises that follow, use residue calculus to prove the displayed identity.

Exercises

(1) $\int_0^\infty \frac{dx}{1+x^n} = \frac{\pi/n}{\sin(\pi/n)}$
[Hint: try a wedge of radius R and angle $2\pi/n$]

(2) $\int_{-\infty}^\infty \frac{x-1}{x^5-1} dx = \frac{4\pi}{5} \sin\frac{2\pi}{5}$

(3) $\int_{-\infty}^\infty \frac{\cos x}{x^2+a^2} dx = \frac{\pi e^{-a}}{a}$, $a \in \mathbb{R}$

(4) $\int_{-\infty}^\infty \frac{\cos x}{(x^2+a^2)^2} dx = \frac{\pi(1+a)}{2a^3 e^a}$, $a \in \mathbb{R}_{>0}$

(5) $\int_{-\infty}^\infty \frac{\cos x}{e^x+e^{-x}} dx = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}$
[Hint: rectangle with vertices $\pm R$, $\pm R + \pi i$]

(6) $\int_0^\infty \frac{(\log x)^2}{1+x^2} dx = \frac{\pi^3}{8}$
[Hint: upper semicircle with a bump avoiding 0]

(7) $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx = -\frac{\pi}{4}$

(8) $\int_0^\pi \frac{1}{1+\sin^2\theta} d\theta = \frac{\pi}{\sqrt{2}}$

(9) $\int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2} d\theta = \frac{2\pi a}{(a^2-b^2)^{3/2}}$, for $0 < b < a$.