III.I. Integral transforms

We are interested in these mainly as a source of integrals on $\mathbb{R}$ for which residue theory is useful. In general, a "transform" is just a $\mathbb{C}$-linear transformation of $\mathbb{C}$-valued functions on the real line:

$$(Tf)(k) := \int f(t)G(k,t)dt.$$

§ **Three famous types.**

(i) Fourier: $\hat{f}(k) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)e^{-\mathbf{i}kx}dx$

(ii) Laplace: $(\mathcal{L}f)(k) := \displaystyle\int_0^{\infty} f(x)e^{-kx}dx$

(iii) Mellin: $(\mathcal{M}f)(k) := \displaystyle\int_0^{\infty} f(x)x^{k-1}dx$

Of course, while k is *a priori* a real variable, sometimes we will want to make these transforms functions on the entire complex plane.

III.I.1. EXAMPLE (The Gamma function). Let

$$f(x) = e^{-x}.$$

Then we define

$$\Gamma(k) := (\mathcal{M}f)(k) = \int_0^{\infty} e^{-x}x^{k-1}dx. \tag{III.I.2}$$

Integrating by parts with $u = e^{-x}$ and $dv = x^{k-1}dx$ yields

$$\Gamma(k) = e^{-x}\frac{x^k}{k}\Big|_0^{\infty} - \int_0^{\infty} \frac{x^k}{k}(-e^{-x})dx = \frac{1}{k}\Gamma(k+1), \tag{III.I.3}$$

which together with $\Gamma(1) = \int_0^{\infty} e^{-x}dx = 1$ implies for all $k \in \mathbb{N}$

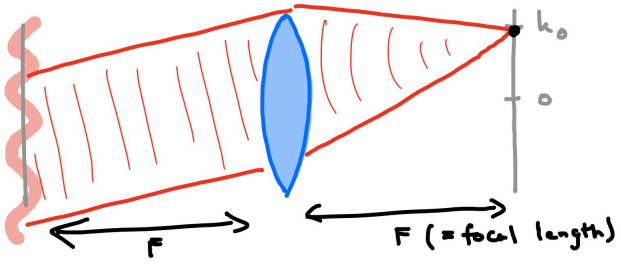
$$\Gamma(k) = (k-1)\Gamma(k-1) = (k-1)(k-2)\Gamma(k-2) = \cdots = (k-1)!.$$

More generally, the integral converges (to a holomorphic function) for $\mathrm{Re}(k) > 0$.

III.I.4. REMARK. We may analytically continue $\Gamma(k)$ to a meromorphic function on all of $\mathbb{C}$ using (III.I.3) or the "functional equation" $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ we'll prove in Complex Analysis II.

§ Fourier transforms.

First, a bit of intuition. Recall that $\hat{f}(k)$ quantifies the "infinitesimal amplitude" of a frequency k in $f(x)$. One can think about this in terms of lenses



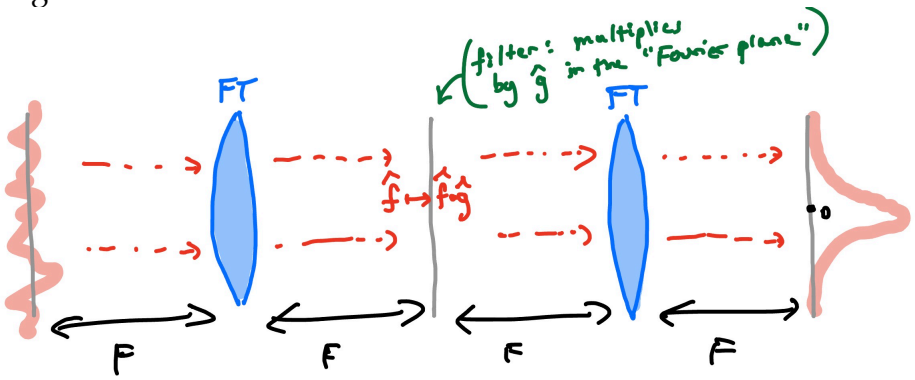
where a pure frequency input e^{ik_0x} on the left corresponds to a tilted wavefront, with output an infinite spike on the right:

$$\int_{-\infty}^{\infty} e^{i(k_0-k)x}dx = \text{“}\delta(k-k_0)\text{”}$$

If we Fourier transform a function $f(x)$ and multiply with the Fourier transform of a second function $g(x)$, and then Fourier transform the result, we obtain the **convolution**

$$\widehat{(\hat{f}\cdot\hat{g})}(y) = (f*g)(y) := \int_{-\infty}^{\infty} f(x)g(y-x)dx. \tag{III.I.5}$$

Informally speaking, the closer f and g are to being the same function, the more $f*g$ is like a spike at $y=0$. In optics, this is used for image detection:

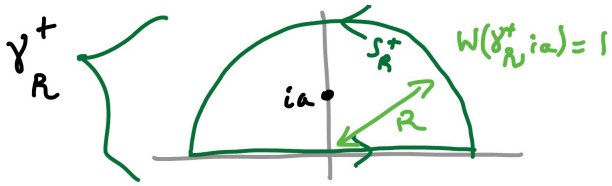


Also, from the "frequency" intuition: a more “concentrated” function should have higher frequencies, so have more “spread-out” FT. We'll test this hypothesis below.

III.I.6. EXAMPLE. We compute

$$I(k) = \int_{-\infty}^{\infty} \frac{e^{\mathbf{i}kx}}{a^2+x^2}dx$$

for $k \in \mathbb{R}$ and $a > 0$. For $k \geq 0$ use

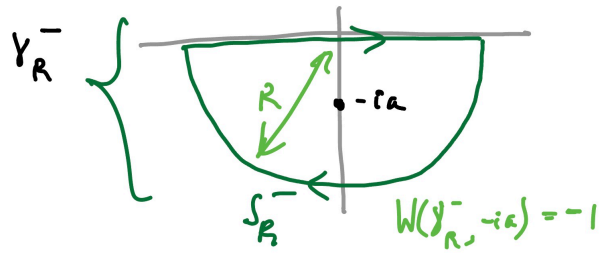


and the bound $\left|\int_{S_R^+} \frac{e^{\mathbf{i}kz}}{a^2+z^2}dz\right| \leq \frac{\pi R}{R^2-a^2} \underset{R\to\infty}{\longrightarrow} 0$ to conclude that

$$I(k) = \lim_{R\to\infty} \int_{\gamma_R^+} \frac{e^{\mathbf{i}kz}}{a^2+z^2}dz = 2\pi\mathbf{i}\mathrm{Res}_{\mathbf{i}a}\left(\frac{e^{\mathbf{i}kz}/(z+\mathbf{i}a)}{z-\mathbf{i}a}\right)$$

$$= 2\pi\mathbf{i}\cdot\frac{e^{\mathbf{i}k(\mathbf{i}a)}}{\mathbf{i}a+\mathbf{i}a} = \frac{\pi}{a}e^{-ak} = \frac{\pi}{a}e^{-|k|a}$$

For $k \leq 0$ use



to get

$$I(k) = -2\pi\mathbf{i}\mathrm{Res}_{-\mathbf{i}a}\left(\frac{e^{\mathbf{i}kz}/(z-\mathbf{i}a)}{z+\mathbf{i}a}\right)$$

$$= -2\pi\mathbf{i}\frac{e^{\mathbf{i}k(-\mathbf{i}a)}}{-2\mathbf{i}a} = \frac{\pi e^{ka}}{a} = \frac{\pi}{a}e^{-|k|a}.$$

So if $f(x) = \frac{a}{a^2+x^2}$, then for *all* $k \in \mathbb{R}$

$$\hat{f}(k) = \frac{a}{\sqrt{2\pi}}I(-k) = \sqrt{\frac{\pi}{2}}e^{-|k|a}.$$

While the area underneath $y{=}f(x)$ is independent of a, taking a smaller makes the graph "spikier" — so, more high-frequency stuff. Therefore we expect a more spread-out FT, which is indeed the case.

Here is a more general result on Fourier transforms: writing $z = x + \mathbf{i}y$, we have the

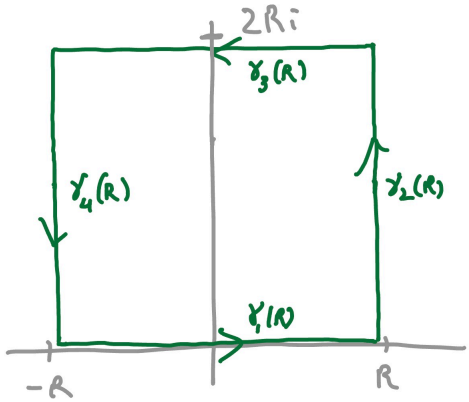
III.I.7. THEOREM. *Assume $f(z) \in \mathscr{M}er(\mathbb{C})$ with only finitely many poles, none of these on $\mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} f(x)e^{\mathbf{i}kx}dx = \begin{cases} 2\pi\mathbf{i}\sum_{y>0}\operatorname{Res}_z(f(z)e^{\mathbf{i}kz}), & k>0 \\ -2\pi\mathbf{i}\sum_{y<0}\operatorname{Res}_z(f(z)e^{\mathbf{i}kz}), & k<0 \end{cases}$$

provided only that $|f(z)| \le \frac{B}{|z|}$ for $|z|$ large.

III.I.8. REMARK. Our last example used $|f(z)| \le \frac{B}{|z|^2}$, as in Example III.H.1. But here we can take advantage of the behavior of $e^{\mathbf{i}kz}$ for $k \neq 0$ to make the "upper part" of the integral go to zero. This upper part won't, however, be a semicircle!

PROOF. This time we take γ_R^+ to be the rectangular path:



Take $k > 0$, $R \gg 0$. Parametrizing $-\gamma_3$ by $t \mapsto t + 2\mathbf{i}R$ gives

$$\left|\int_{-\gamma_3(R)} e^{\mathbf{i}kz}f(z)dz\right| = \left|\int_{-R}^{R} e^{\mathbf{i}k(t+2\mathbf{i}R)}f(t+2\mathbf{i}R)dt\right|$$
$$\le 2\cancel{R}\cdot e^{-2kR}\cdot\frac{B}{\cancel{2R}} = Be^{-2kR} \xrightarrow[R\to\infty]{} 0.$$

Next, parametrizing γ_2 by $t \mapsto R + \mathbf{i}t$, we have

$$\left|\int_{\gamma_2(R)} e^{\mathbf{i}kz} f(z)dz\right| = \left|\int_0^{2R} e^{\mathbf{i}k(R+\mathbf{i}t)} f(R+\mathbf{i}t)\mathbf{i}dt\right|$$
$$\leq \int_0^{2R} e^{-kt}\frac{B}{R}dt = \frac{B}{kR}(1-e^{-2kR}) \underset{R\to\infty}{\longrightarrow} 0$$

Same story with $\int \gamma_4(R)$. So by the residue theorem

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)e^{\mathbf{i}kx}dx = \lim_{R\to\infty}\int_{\gamma_R^+} f(z)e^{\mathbf{i}kz}dz$$
$$= 2\pi\mathbf{i}\sum_{y>0}\mathrm{Res}_z(f(z)e^{\mathbf{i}kz}).$$

The $k < 0$ case is similar, with γ_R^+ replaced by γ_R^-. □

§ **Mellin transforms.**

III.I.9. EXAMPLE. Let

$$f(x) = \frac{1}{1+x^2}.$$

We will compute $(Mf)(k)$ for $k \in (0,2)$, or equivalently

$$I(\alpha) := \int_0^\infty \frac{x^\alpha dx}{1+x^2} \quad \text{for} \quad -1 < \alpha < 1.$$

For $\alpha = 0$,

$$I(0) = \arctan(\infty) - \arctan(0) = \frac{\pi}{2}.$$

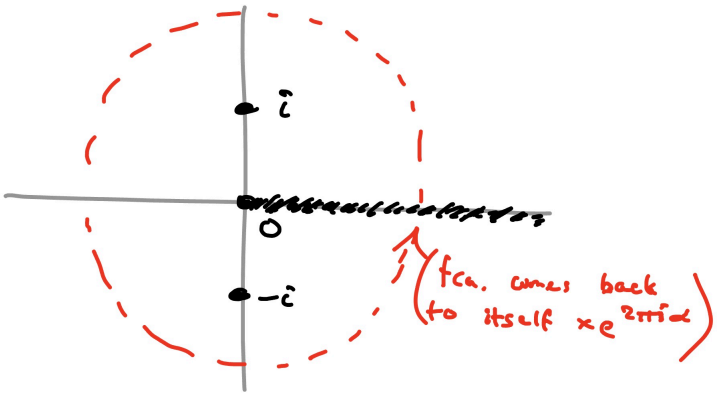
The method we use now works for $\alpha \in (-1,0)\cup(0,1)$. For $\alpha \neq 0$, the integrand (regarded as a function of z)

$$F(z) := \frac{z^\alpha}{1+z^2} = \frac{e^{\alpha\log(z)}}{1+z^2}$$

is not single-valued in the complex plane: write $z = re^{\mathbf{i}\theta}$, let θ run from 0 to 2π; then F doesn't return to itself —

$$F(re^{2\pi\mathbf{i}}) \text{ “=” } \frac{r^\alpha}{1+r^2}e^{2\pi\mathbf{i}\alpha} = F(re^{0\mathbf{i}})\cdot e^{2\pi\mathbf{i}\alpha}.$$

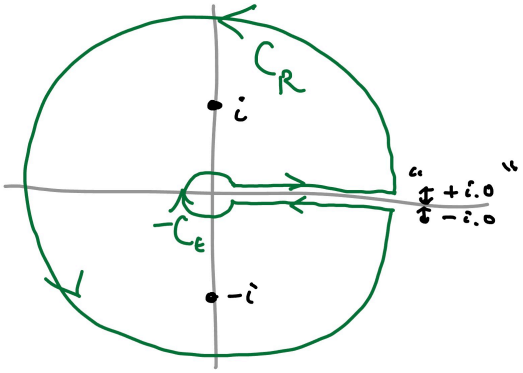
But we *can* regard F as a single-valued function on the cut plane:



More precisely, we have

$$F \in \mathscr{H}\!ol(\mathbb{C} \setminus (\mathbb{R}_{\geq 0} \cup \{-\mathbf{i}, +\mathbf{i}\})).$$

So consider the path



and write

$$\begin{aligned}
J(\alpha) :&= \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \left(\int_{C_R} F dz - \int_{C_\epsilon} F dz + \int_\epsilon^R \underbrace{F(x+\mathbf{i}0)}_{F(x)} dx - \int_\epsilon^R \underbrace{F(x-\mathbf{i}0)}_{F(x)e^{2\pi\mathbf{i}\alpha}} dx \right) \\
&= 2\pi\mathbf{i}\left(\mathrm{Res}_{\mathbf{i}}(F(z)) + \mathrm{Res}_{-\mathbf{i}}(F(z))\right) \quad \text{[by the residue theorem]} \\
&= 2\pi\mathbf{i}\left(\mathrm{Res}_{\mathbf{i}}\left(\frac{e^{\alpha \log z}/(z+\mathbf{i})}{z-\mathbf{i}}\right) + \mathrm{Res}_{-\mathbf{i}}\left(\frac{e^{\alpha \log z}/(z-\mathbf{i})}{z+\mathbf{i}}\right)\right) \\
&= 2\pi\mathbf{i}\left(\frac{e^{\alpha \log \mathbf{i}}}{2\mathbf{i}} + \frac{e^{\alpha \log(-\mathbf{i})}}{-2\mathbf{i}}\right) \quad = \quad 2\pi\mathbf{i}\frac{e^{\alpha\frac{\pi\mathbf{i}}{2}} - e^{\alpha\frac{3\pi\mathbf{i}}{2}}}{2\mathbf{i}} \\
&= 2\pi\mathbf{i}e^{\alpha\pi\mathbf{i}}\frac{e^{-\alpha\pi\mathbf{i}/2} - e^{\alpha\pi\mathbf{i}/2}}{2\mathbf{i}} \quad = \quad -2\pi\mathbf{i}e^{\alpha\pi\mathbf{i}}\sin(\alpha\pi/2).
\end{aligned}$$

On the other hand,[19] we have

$$\left|\int_{C_R} F dz\right| \leq 2\pi R \left\|\frac{z^\alpha}{1+z^2}\right\|_{C_R} \leq 2\pi R \cdot \frac{R^\alpha}{R^2-1} \to 0$$

as $R \to \infty$ and

$$\left|\int_{C_\epsilon} F dz\right| \leq 2\pi\epsilon \left\|\frac{z^\alpha}{1+z^2}\right\|_{C_\epsilon} \leq 2\pi\epsilon \cdot \frac{\epsilon^\alpha}{1-\epsilon^2} \to 0$$

as $\epsilon \to 0$. So

$$\begin{aligned} J(\alpha) &= 0 - 0 + \lim \int_\epsilon^R F dx - e^{2\pi \mathbf{i}\alpha} \lim \int_\epsilon^R F dx \\ &= (1 - e^{2\pi\mathbf{i}\alpha}) I(\alpha) \\ &= 2\mathbf{i} e^{\pi\mathbf{i}\alpha} \frac{e^{-\pi\mathbf{i}\alpha} - e^{\pi\mathbf{i}\alpha}}{2\mathbf{i}} I(\alpha) \\ &= -2\mathbf{i} e^{\alpha\pi\mathbf{i}} \sin(\pi\alpha) I(\alpha), \end{aligned}$$

and $-2\pi\mathbf{i} e^{\alpha\pi\mathbf{i}} \sin\left(\frac{\alpha\pi}{2}\right) = J(\alpha) = -2\mathbf{i} e^{\alpha\pi\mathbf{i}} \sin(\pi\alpha) I(\alpha)$ now gives

$$I(\alpha) = \frac{\pi \sin(\frac{\alpha\pi}{2})}{\sin(\alpha\pi)}.$$

More generally, Mellin transforms are almost always computed using the residues of the function $f(z)z^{k-1}$ in the complex plane (except at $z = 0$), and by the same technique.

§ Application to modular forms.

We'll jump into some deep water here; related stuff will be done more carefully in Complex Analysis II when we look at the Gamma and Riemann zeta functions.

Let $g \in \mathscr{Hol}(\mathfrak{H})$ satisfy the "automorphy property"

$$(*)_k \qquad \boxed{g(f_\gamma(\tau)) = (c\tau + d)^k g(\tau) \qquad \left(\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \gamma \in \mathrm{SL}_2(\mathbb{Z})\right)}$$

where $f_\gamma(\tau) = \frac{a\tau+b}{c\tau+d} \colon \mathfrak{H} \xrightarrow{\cong} \mathfrak{H}$ is the familiar FLT.

[19] noting $\alpha \in \mathbb{R}$ on C_R, and using $|z^\alpha| = |e^{\alpha \log(z)}| = |e^{\alpha \log R}||e^{\mathbf{i}\alpha\theta}| = |e^{\alpha \log R}| = R^\alpha$; and similarly $|z^\alpha| = \epsilon^\alpha$ on C_ϵ. We are also using $\alpha + 1 < 2$ for the first "$\to 0$" and $\alpha + 1 > 0$ for the second.

Since $(*)_k$ is compatible with the group structure and $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbb{Z})$ (see the Exercises), it is actually enough to check

(A) $(*)_k$ for $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$: $\quad g(\tau+1) = g(\tau)$
(B) $(*)_k$ for $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$: $\quad g(-1/\tau) = \tau^k g(\tau)$

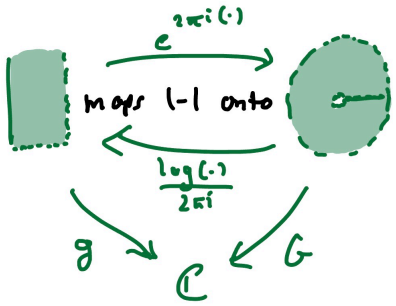
From (A), we have

$$g(\tau) = G(e^{2\pi \mathbf{i}\tau}) \tag{III.I.10}$$

for some $G \in \mathscr{Hol}(D_1^*)$:

$$\tau \mapsto e^{2\pi\mathbf{i}\tau} =: q \quad \text{maps} \quad \mathfrak{H}/\langle\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle \xrightarrow{\cong} D_1^*$$

since $e^{2\pi\mathbf{i}(u+\mathbf{i}v)} = e^{2\pi\mathbf{i}u}e^{-2\pi v}$. That is, for each vertical strip (of the form $\mathrm{Re}(\tau) \in [n, n+1)$, $\mathrm{Im}(\tau) > 0$)



Using Laurent series,

$$G(q) = \sum_{n=-\infty}^{\infty} a_n q^n \implies g(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi\mathbf{i}n\tau}.$$

Now consider the conditions

(1) 0 is a removable singularity of G
($\iff$ $a_n = 0$ for $n < 0$)
($\iff$ g is "bounded at ∞", i.e. on $\{\tau | \, \mathrm{Im}(\tau) \geq a\}$ for $a > 0$)

(2) 0 is a removable singularity and $G(0) = 0$
($\iff$ $a_n = 0$ for $n \leq 0$).

III.I.11. DEFINITION. Suppose $g \in \mathscr{Hol}(\mathfrak{H})$ satisfies $(*)_k$ (or equivalently (A) and (B)). We call g a **modular form** of weight k if (1) holds, and a **cusp form** of weight k if (2) holds.

Suppose then that

$$g(\tau) = \sum_{n\geq 1} c_n e^{2\pi \mathbf{i} n\tau}$$

is a cusp form of weight k, and put

$$\tilde{g}(z) := g(\mathbf{i}z) \in \mathscr{H}\mathscr{o}\ell(U_0)$$

where U_0 := right half-plane. Computing "formally," the Mellin transform

$$\begin{aligned}
(\mathcal{M}\tilde{g})(s) &= \int_0^\infty \tilde{g}(x)x^{s-1}dx \\
&= \int_0^\infty \sum_{n\geq 1} c_n e^{2\pi \mathbf{i} n(\mathbf{i}x)} x^{s-1}dx \\
&= \sum_{n\geq 1} c_n \int_0^\infty e^{-2\pi nx} x^{s-1}dx \\
&= \sum_{n\geq 1} c_n \int_0^\infty e^{-t}(2\pi n)^{-s} t^{s-1}dt \qquad [t = 2\pi nx] \\
&= (2\pi)^{-s}\Gamma(s) \sum_{n\geq 1} \frac{c_n}{n^s} \\
&=: (2\pi)^{-s}\Gamma(s)L(s,g) \\
&=: L^*(s,g).
\end{aligned}$$

Here $L(s,g)$ is called the **L-function** of the cusp form g.

So, for which s is this valid? Consider the function

$$\varphi(\tau) := |g(\tau)|\mathrm{Im}(\tau)^{k/2} \in \mathcal{C}^\infty(\mathfrak{H}).$$

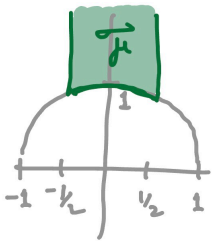
This satisfies $\varphi(\tau+1) = \varphi(\tau)$ and

$$\begin{aligned}
\varphi\left(-\tfrac{1}{\tau}\right) &= \left|g\left(-\tfrac{1}{\tau}\right)\right| \left(\frac{\mathrm{Im}(\tau)}{|\tau|^2}\right)^{k/2} \\
&= \cancel{|\tau|^k}|g(\tau)|\mathrm{Im}(\tau)^{k/2}\cancel{|\tau|^{-k}} = \varphi(\tau).
\end{aligned}$$

So φ is $\mathrm{SL}_2(\mathbb{Z})$-invariant.

III.I.12. LEMMA. *φ is bounded on $\mathfrak{H}$.*

SKETCH. $|G(q)| \le C|q|$ for $|q| \le \frac{1}{2} \implies |g(\tau)| \le Ce^{-2\pi \mathrm{Im}(\tau)}$ for $\mathrm{Im}(\tau) \ge \frac{\log 2}{2\pi} \implies \varphi(\tau) \le Ce^{-2\pi \mathrm{Im}(\tau)} \mathrm{Im}(\tau)^{k/2} \to 0$ (as $\mathrm{Im}(\tau) \to \infty$). We'll learn later about the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, which is $\mathcal{F}$:



Clearly φ is bounded on $\mathcal{F}$; therefore, by $\mathrm{SL}_2(\mathbb{Z})$-invariance, it is bounded on $\mathfrak{H}$. □

Writing $B := \|\varphi\|_{\mathfrak{H}}$, this leads to the next

III.I.13. LEMMA. *$|c_n| \le B_0 \cdot n^{k/2}$, where $B_0 = Be^{2\pi}$.*

SKETCH. Recall that $G(q) = \sum_{n \ge 1} c_n q^n$, so that (taking $0 < r < 1$ and $b = \frac{\log r}{-2\pi} > 0$)

$$c_n = \frac{1}{2\pi \mathbf{i}} \oint_{\partial D_r} \frac{G(q)dq}{q^{n+1}} = \frac{1}{\cancel{2\pi \mathbf{i}}} \int_{-1/2+\mathbf{i}b}^{1/2+\mathbf{i}b} \frac{g(\tau)\cancel{2\pi \mathbf{i}} \cancel{e^{2\pi \mathbf{i}\tau}} d\tau}{e^{2\pi \mathbf{i}(n+\cancel{1})\tau}}$$
$$= \int_{-1/2+\mathbf{i}b}^{1/2+\mathbf{i}b} g(\tau) e^{-2\pi \mathbf{i} n \tau} . d\tau$$

Using Lemma III.I.12, this yields $|c_n| \le \|g(\tau)\| e^{2\pi n b} \le Bb^{-k/2} e^{2\pi n b}$ for all $b > 0$, and taking $b = \frac{1}{n}$ gives $|c_n| \le B \cdot n^{k/2} e^{2\pi} = B_0 n^{k/2}$. □

III.I.14. PROPOSITION. *Let g be a cusp form of weight k. Then the series defining $L(s, g)$ converges absolutely for $\mathrm{Re}(s) > \frac{k}{2} + 1$, hence the "formal" computation above is valid.*

SKETCH. By Lemma III.I.13, we have

$$\frac{|c_n|}{n^s} \le B_0 \cdot \frac{1}{n^{s-k/2}}.$$

Apply integral test; then notice that in the $\int \sum \leftrightarrow \sum \int$ swap, all functions/terms are > 0. □

III.I.15. THEOREM (Hecke). *$L(s,g)$ extends to an entire function, and may be described in* $\mathrm{Re}(s) < \frac{k}{2} - 1$ *by means of the functional equation*

$$L^*(s,g) = (-1)^{k/2} L^*(k-s,g).$$

PROOF. Write $\tau = u + \mathbf{i}v$, $u = 0$. Then

(a) $g\left(-\frac{1}{\tau}\right) = \tau^k g(\tau)$ becomes $g\left(\frac{\mathbf{i}}{v}\right) = \mathbf{i}^k v^k g(\mathbf{i}v)$.

(b) $|g(\tau)| \le Ce^{-2\pi v}$ for $v \ge \frac{1}{2\pi}\log 2 \implies \int_1^\infty g(\mathbf{i}v) v^{s-1} dv$ converges for all s (to an entire function).

Now, for $\mathrm{Re}(s) > \frac{k}{2} + 1$,

$$L^*(s,g) = \int_0^1 g(\mathbf{i}v) v^{s-1} dv + \int_1^\infty g(\mathbf{i}v) v^{s-1} dv.$$

Replacing v by $\frac{1}{v}$ in the first integral yields $-\int_1^\infty g(\frac{\mathbf{i}}{v}) v^{1-s} \frac{dv}{v^2}$, which by (a) becomes $\mathbf{i}^k \int_1^\infty v^{k-s-1} g(\mathbf{i}v) dv$. In view of (b), we therefore have

$$L^*(s,g) = \int_1^\infty g(\mathbf{i}v)\{v^{s-1} + \mathbf{i}^k v^{k-s-1}\} dv \in \mathscr{Hol}(\mathbb{C}). \tag{III.I.16}$$

At this point it is clear that $L(s,g) = \frac{(2\pi)^s}{\Gamma(s)} L^*(s,g)$ is entire, since $\Gamma(s)$ is nowhere 0.[20] Also, replacing s by $k-s$ yields

$$\mathbf{i}^k L^*(k-s,g) = \mathbf{i}^k \int_1^\infty g(\mathbf{i}v)\{v^{k-s-1} + \mathbf{i}^k v^{s-1}\} dv,$$

which equals (III.I.16) if k is even. But there are no cusp forms of odd weight: if $\gamma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, then $f_\gamma(\tau) = \tau$ and $g(\tau) = g(f_\gamma(\tau)) = (-1)^k g(\tau)$, which forces $g \equiv 0$ if k is odd. □

III.I.17. EXAMPLE. When we discuss infinite products we'll meet Weierstrass's **modular discriminant**

$$\Delta(\tau) := q \prod_{n>0} (1-q^n)^{24} = \sum c_n q^n = q - 24q^2 + 252q^3 - \dots.$$

The reason for the name is that it arises in the context of modular parametrizations of elliptic curves $y^2 = 4x^3 - g_2 x - g_3$, and then $\Delta = g_2^3 - 27 g_3^2$ is the discriminant of the cubic polynomial in x. The c_n here are usually denoted $\tau(n)$ (obviously a conflict of notation at

[20] We have not proved this, but it is easy to see from Remark III.I.4. Properties of $\Gamma(s)$ and related functions will be covered in more detail in Complex Analysis II.

present), and called the **Ramanujan tau function**. These objects have the following properties:

- Δ is a cusp form of weight 12.
- $c_n c_m = c_{nm}$ if $\gcd(n,m) = 1$.
- $|c_p| \leq 2p^{1/2}$ for p prime (from above, know at least $|c_n| \leq B_0 n^6$).

III.I.18. CONJECTURE (Ramanujan). *The zeroes of $L(s,\Delta)$ in the critical strip $\frac{11}{2} < \mathrm{Re}(s) < \frac{13}{2}$ all lie on $\mathrm{Re}(s) = 6$.*

Remind you of something? Only, I suppose this one isn't worth $1,000,000.

Exercises

(1) Let f be a continuous function, and suppose $\int_0^\infty f(x)\, x^a \frac{dx}{x}$ is absolutely convergent. Show that it is equal to $\int_{-\infty}^{\infty} f(e^t)\, e^{at} dt$, hence that a Mellin transform is essentially a Fourier transform up to a change of variable.

(2) Recall that the hyperbolic cosine is $\cosh(x) = \frac{e^x + e^{-x}}{2}$, with inverse $\mathrm{sech}(x) = \frac{1}{\cosh(x)}$ the hyperbolic secant. Use residue theory to find the Fourier transform of $\mathrm{sech}(x)$.

(3) Let $f(x) = \frac{1}{1+x^3}$. Compute the Mellin transform $(\mathcal{M}f)(a)$ for $a \in (0,3)$.

(4) Show that the rule $g(f_\gamma(\tau)) = (c\tau + d)^k g(\tau)$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, satisfied by a modular form of weight k, is compatible with the group structure — i.e., if it is satisfied for γ_1 and γ_2 then it is satisfied for $\gamma_1\gamma_2$.

(5) Show that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ generate $\mathrm{SL}_2(\mathbb{Z})$.