IV. Harmonic functions

IV.A. Laplace's equation

§ **Motivation: the Dirichlet problem.**

Harmonic functions crop up in thermodynamics, electrostatics, fluid mechanics, and many other "natural" settings. Natural in flatland, since we'll be sticking to 2 (real) spatial dimensions here.

IV.A.1. EXAMPLE. The **heat equation** describes how the rate of change of temperature T at a point depends on nearby temperature

$$\frac{\partial T}{\partial t} = k \cdot \Delta T$$

where $T = T(t, x, y)$,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is the **Laplacian** operator, and $k =$ thermal diffusivity of the material. Hence, Laplace's equation

$$\Delta T \equiv 0$$

holds when T is stable in time.

Obviously, a constant solves this equation; but physical intuition says that if you place a heat source[1] at the boundary of a region U, after a long time elapses T should stabilize, yielding a solution $T(x, y)$ of $\Delta T \equiv 0$. Modeling this heat source by $T_0 \colon \partial U \to \mathbb{R}$, this stable temperature distribution should be a continuous function $T \colon \overline{U} \to \mathbb{R}$ with $T|_{\partial U} \equiv T_0$ and $\Delta T \equiv 0$ on U. (For the latter to make sense, we need to have T at least $\mathcal{C}^2$ on U.)

More generally, let $U \subset \mathbb{C}$ be an open set.

[1] or cold source (e.g., children)

IV.A.2. DEFINITION. A function $u\colon U \to \mathbb{R}$ is **harmonic** if $u \in \mathcal{C}^2(U)$ and $\Delta u \equiv 0$. In this case we write $u \in \mathcal{H}(U)$. Note that u is a *real-valued* function of a *complex* variable (or two real ones).

IV.A.3. DIRICHLET BOUNDARY VALUE PROBLEM. *Given $u_0\colon \partial U \to \mathbb{R}$ piecewise $\mathcal{C}^0$, when does there exist a unique (piecewise $\mathcal{C}^0$) $u\colon \overline{U} \to \mathbb{R}$ harmonic on U and equal to u_0 on ∂U?*

At this level of generality, the answer is NO:

- It's only reasonable to ask for $u|_{\partial U} = u_0$ where u_0 is continuous — or at least, "interpolates the discontinuity" as in Fourier series:



- Existence of u fails too for some cases of non-simply-connected U and *continuous* u_0, e.g. for $U = D_1^*$.
- IV.A.3 is always soluble for, say, u_0 continuous and U simply connected.

IV.A.4. EXAMPLE (Electrostatics/voltage). Place an electric charge on ∂U; then the induced electric potential function "solves Dirichlet." Proof by nature! If you add electric charges au_0 and bv_0 on the boundary, then the solutions add. (Principle of superposition.) This illustrates that linear combinations of harmonic functions are harmonic. Products are *not*, e.g. $x \in \mathcal{H}(\mathbb{C})$ but $x^2 \notin \mathcal{H}(\mathbb{C})$.

Our first step below is to prove a relation between harmonic and holomorphic functions that will bring to bear all of the techniques of complex analysis.

§ Harmonic conjugates.

Let $f \in \mathscr{Hol}(U)$, $f = u + \mathbf{i}v$. Since f is infinitely complex-differentiable, u and v are $\mathcal{C}^\infty$. In fact:

IV.A.5. PROPOSITION. *The real and imaginary parts of a holomorphic (or antiholomorphic) function are harmonic.*

PROOF. Use the Cauchy-Riemann equations:

$$\begin{array}{ccc} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} & & \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \\ \frac{\partial}{\partial x}\downarrow & & \downarrow\frac{\partial}{\partial y} \\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial}{\partial y}v & = & \frac{\partial}{\partial y}\frac{\partial}{\partial x}v = -\frac{\partial^2 u}{\partial y^2} \end{array}$$

$\implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \implies u \in \mathcal{H}(U)$. The version for antiholomorphic functions follows since they are just the complex conjugates of holomorphic functions. □

IV.A.6. EXAMPLES.

- on $\mathbb{C}$: $ax + by + c$, $e^x \cos y$, and $x^2 - y^2$;
- on $\mathbb{C}^*$: $\frac{1}{2}\log(x^2 + y^2) = \log|z|$, and $\frac{x}{x^2+y^2}$.

Conversely, when might a given harmonic function be the real part of a holomorphic function? It's fairly clear this won't work for $\log|z|$ on D_1^*, since $\log(z)$ (of which it is the real part) isn't a well-defined function. One suspects here that the problem is that D_1^* isn't simply connected. So we shall

assume that U is simply connected and $u \in \mathcal{H}(U)$.

Next recall that:

- $\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - \mathbf{i}\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right)$
- f is holomorphic $\iff \frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial f}{\partial z} = \frac{df}{dz} (=: f')$

The first bullet yields

$$\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \frac{1}{4}\left(\frac{\partial}{\partial x} + \mathbf{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \mathbf{i}\frac{\partial}{\partial y}\right) = \frac{1}{4}\Delta. \tag{IV.A.7}$$

Set $g := 2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} - \mathbf{i}\frac{\partial u}{\partial y}$, so that $\Delta u = 0 \implies$

$$\frac{\partial g}{\partial \bar{z}} = 2\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}u = \frac{1}{2}\Delta u = 0 \quad \implies \quad g \in \mathscr{Hol}(U).$$

Since U is simply connected, g has a primitive $f \in \mathscr{Hol}(U)$ (with $g = f'$). Write $f = u_1 + \mathbf{i}v_1$. Then using $\frac{\partial v_1}{\partial y} = \frac{\partial u_1}{\partial x}$ and $\frac{\partial v_1}{\partial x} = -\frac{\partial u_1}{\partial y}$

$$\begin{aligned}\frac{\partial u}{\partial x} - \mathbf{i}\frac{\partial u}{\partial y} = g = f' = \frac{\partial f}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x}(u_1 + \mathbf{i}v_1) - \mathbf{i}\frac{\partial}{\partial y}(u_1 + \mathbf{i}v_1)\right) \\ &= \frac{\partial u_1}{\partial x} + \mathbf{i}\left(-\frac{\partial u_1}{\partial y}\right) = \frac{\partial u_1}{\partial x} - \mathbf{i}\frac{\partial u_1}{\partial y}\end{aligned}$$

$$\implies \frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x},\ \frac{\partial u}{\partial y} = \frac{\partial u_1}{\partial y} \implies u_1 = u + C \quad \text{(real constant)}.$$

Taking $F := f - C$, we have $\mathrm{Re}(F) \equiv u$.

There is also the question of the uniqueness of this F. If $F_0 \in \mathscr{Hol}(U)$ with $\mathrm{Re}(F_0) \equiv u$ then $\mathrm{Re}(F - F_0) \equiv 0$ on U, i.e. $F - F_0$ is a holomorphic function mapping to a straight line. This contradicts the open mapping theorem unless $F - F_0$ is a (necessarily imaginary) constant. We have proved the

IV.A.8. PROPOSITION-DEFINITION. *A harmonic function u on a simply connected open set is the real part of a holomorphic function there (unique up to an imaginary constant), whose imaginary part v is called the* ***harmonic conjugate*** *of u. (Its level curves are perpendicular to those of u.)*

IV.A.9. EXAMPLES. Here are the harmonic conjugates of the harmonic functions in IV.A.6. Note that they are defined up to a real constant, which I don't include.

- on $\mathbb{C}$: $-bx + ay$, $e^x \sin y$, and $2xy$;
- first one on $\mathbb{C}^*$: $\arctan(y/x) = \arg(z)$, which isn't well defined[2] unless we insert a cut (to make U simply connected);
- second one on $\mathbb{C}^*$: $\frac{-y}{x^2+y^2}$.

[2] even though $\frac{1}{2}\log(x^2 + y^2)$ *was*! This illustrates the necessity of the simple-connectivity hypothesis.

As to the etymology of the term "harmonic", consider any $u \in \mathcal{H}(D_{1+\epsilon})$, which by IV.A.8 has to be the real part of some holomorphic function $\sum_{n\geq 0}(a_n + \mathbf{i}b_n)z^n$. Restricting to the unit circle C_1 gives

$$\begin{aligned} u(e^{\mathbf{i}\theta}) &= \operatorname{Re}\left(\sum_{n\geq 0}(a_n + \mathbf{i}b_n)(\cos(n\theta) + \mathbf{i}\sin(n\theta))\right) \\ &= \sum_{n\geq 0}(a_n\cos(n\theta) - b_n\sin(n\theta)) \end{aligned}$$

The $\cos(n\theta)$ and $\sin(n\theta)$ here bring to mind the "harmonic overtones" on a vibrating string. This also suggests a relation between Fourier series and the Dirichlet problem for D_1, which I'll make more precise when we discuss the latter.

§ Basic properties of harmonic functions.

We start by picking some *really* low-hanging fruit.

IV.A.10. COROLLARY. *Harmonic functions are $\mathcal{C}^\infty$ (i.e., smooth).*

PROOF. Any harmonic function takes the form $u = \operatorname{Re}(f)$ with f holomorphic, which makes

$$f' = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \mathbf{i}\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - \mathbf{i}\frac{\partial u}{\partial y}$$

holomorphic, hence $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ harmonic. Repeat this argument for both of them, *ad infinitum*. □

IV.A.11. COROLLARY. *Given $u\colon V \to \mathbb{R}$ harmonic and $g\colon U \to V$ holomorphic (or antiholomorphic), $u \circ g\colon U \to \mathbb{R}$ is harmonic.*

PROOF. Given $u = \operatorname{Re}(f)$, f holomorphic. If g is holomorphic, so is $f \circ g$. If g is antiholomorphic, so is $f \circ g$ since $\frac{\partial}{\partial z}(f \circ g) = \frac{\partial f}{\partial z}\cancelto{0}{\frac{\partial \overline{g}}{\partial z}} + \cancelto{0}{\frac{\partial f}{\partial \overline{z}}}\frac{\partial \overline{g}}{\partial z} = 0$. In either case, $\operatorname{Re}(f \circ g) = u \circ g \in \mathcal{H}(U)$, done. □

IV.A.12. COROLLARY (Maximum Principle). *Let $U \subset \mathbb{C}$ be a region, and $u \in \mathcal{C}^0(\overline{U})$ with $u|_U \in \mathcal{H}(U)$. If $u(z_0) \geq u(z)\ \forall z \in \overline{U}$, then $z_0 \in \partial U$ or u is constant.*

PROOF. If U is simply connected, $\exists f \in \mathscr{Hol}(U)$ with $\mathrm{Re}(f) = u$, and $|e^f| = |e^u e^{\mathbf{i}v}| = e^u$. Apply the maximum modulus principle to e^f.

More generally, U is a union of (finitely many) simply connected open sets and the result follows easily from the simply connected case. □

IV.A.13. REMARK. One gets also a "minimum principle" (replacing "$\geq$" by "$\leq$" in Corollary IV.A.12) by replacing e^f by e^{-f} in the proof.



So unless u is constant, the values taken by u on U lie strictly between $\max_{z\in\partial U} u(z)$ and $\min_{z\in\partial U} u(z)$ — resembling a soap film (not entirely correct, since graphs of harmonic functions are not minimal surfaces).

Recall that $\Delta u \equiv 0$ means heuristically that "u is a time-stable temperature function." So it should seem reasonable that at any point, its value is the *average* of nearby values:

IV.A.14. MEAN VALUE THEOREM. $u \in \mathcal{H}(U), U \supset \overline{D}(z_0, r) \implies$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{\mathbf{i}\theta})d\theta$$

PROOF. $\exists f \in \mathscr{Hol}(D(z_0, r+\epsilon))$ with $\mathrm{Re}(f) = u$. By the Cauchy integral formula,

$$u(z_0) + \mathbf{i}v(z_0) = f(z_0) = \frac{1}{2\pi\mathbf{i}}\int_{\partial D(z_0,r)} \frac{f(w)}{w - z_0} dw.$$

Substituting $w = z_0 + re^{\mathbf{i}\theta}$, $dw = \mathbf{i}re^{\mathbf{i}\theta}d\theta$ makes this

$$= \frac{1}{2\pi\mathbf{i}}\int_0^{2\pi} \frac{f(z_0 + re^{\mathbf{i}\theta})}{re^{\mathbf{i}\theta}} \mathbf{i}re^{\mathbf{i}\theta}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{\mathbf{i}\theta})d\theta + \mathbf{i}(\dots).$$

Taking real parts gives the result. □

In studying holomorphic functions we used Cauchy to prove Liouville's theorem. To prove a version for harmonic functions, one

could try to show that

$$u \text{ bounded on } \mathcal{H}(\mathbb{C}) \implies \text{harmonic conjugate is also bounded.}$$

But in fact we get a stronger result by arguing as follows.

IV.A.15. COROLLARY. *$u \in \mathcal{H}(\mathbb{C})$ bounded above* or *below $\implies$ u is constant.*

PROOF 1. We know there exists $f \in \mathscr{Hol}(\mathbb{C})$ with $\mathrm{Re}(f) = u$. So e^f and e^{-f} are holomorphic, with moduli e^u resp. e^{-u}. By assumption, one of these is bounded (above); now apply Liouville. □

PROOF 2. Here is a direct argument, assuming $u \geq 0$. Using $dA = r\,dr\,d\theta$, the Mean Value Theorem gives

$$u(z_0) - u(0) = \frac{1}{\pi R^2}\left(\iint_{D(z_0,R)} u\,dA - \iint_{D(0,R)} u\,dA\right).$$

As the non-overlapping parts of $D(z_0, R)$ and $D(0, R)$ are contained in $\mathcal{A} := \{R - |z_0| < |z| < R + |z_0|\}$,

$$|u(z_0) - u(0)| \leq \frac{1}{\pi R^2}\iint_{\mathcal{A}} u\,dA$$

which by the Mean Value Theorem again

$$\begin{aligned} &= \frac{1}{\pi R^2}\{-\pi(R - |z_0|)^2 u(0) + \pi(R + |z_0|)^2 u(0)\} \\ &= u(0) \cdot \frac{4R|z_0|}{R^2} \end{aligned}$$

which goes to 0 as $R \to \infty$. Thus $u(z_0) = u(0)$ for any z_0. □

Exercises

(1) Find an analytic function whose real part is: (a) $3x^2y - y^3$; (b) $x - xy$; (c) $\frac{y}{(x-t)^2+y^2}$ ($t \in \mathbb{R}$ fixed). In each case, compute the harmonic conjugate.

(2) Denoting by $\mathcal{A}$ the annulus $\{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$, prove the "Hadamard three-circle theorem": given $f \in C^0(\mathcal{A})$ with $f|_{\mathcal{A}} \in Hol(\mathcal{A})$, we have

$$M(r) \leq M(r_1)^{\alpha} M(r_2)^{1-\alpha}$$

where $\alpha = \frac{\log(r_2/r)}{\log(r_2/r_1)}$ and $M(r) := \|f\|_{C_r}$. Discuss cases of equality. [Hint: apply the maximum principle (for harmonic functions) to a linear combination of $\log|f(z)|$ and $\log|z|$.]