IV.B. The Poisson formula

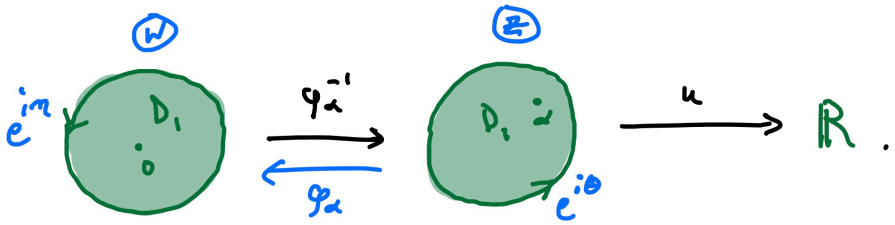
§ Harmonic functions on D_1.

Let u be a harmonic function on some open neighborhood of a closed disk $\overline{D}(z_0, r)$. The Mean Value Theorem lets us recover $u(z_0)$ from the values of u on the boundary. But what about $u(\alpha)$ for $\alpha \in D(z_0, r)$ *different* from z_0?

For simplicity set $z_0 = 0$, $r = 1$,[3] and consider $u \in \mathcal{H}(\overline{D}_1)$ (i.e., u is harmonic on an open set $U \supset \overline{D}_1$). Recall

$$\varphi_\alpha(z) := \frac{z - \alpha}{1 - \bar{\alpha} z}$$

and consider the composition



Note that φ_α^{-1} holo. $\implies$ $u \circ \varphi_\alpha^{-1}$ harmonic $\implies$

$$\begin{aligned} u(\alpha) = (u \circ \varphi_\alpha^{-1})(0) &= \frac{1}{2\pi} \int_0^{2\pi} u(\varphi_\alpha^{-1}(e^{\mathbf{i}\eta}))d\eta \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(e^{\mathbf{i}\theta}) \mathrm{darg}\varphi_\alpha(z) \end{aligned}$$

by the MVT. To make the correspondence between $\eta = \arg(w)$ and $\theta = \arg(z)$ on ∂D_1 explicit, write (with $z = e^{\mathbf{i}\theta}$)

$$\begin{aligned} \mathrm{darg}\varphi_\alpha(z) &= -\mathbf{i}\mathrm{dlog}\varphi_\alpha(z) = -\mathbf{i}\frac{\varphi_\alpha'(z)}{\varphi_\alpha(z)}dz \\ &= -\mathbf{i}\frac{(1 - |\alpha|^2)/(1 - \bar{\alpha}z)^2}{(z - \alpha)/(1 - \bar{\alpha}z)}dz = \frac{1 - |\alpha|^2}{(z - \alpha)(\bar{z} - \bar{\alpha})}(-\mathbf{i}\bar{z}dz) \\ &= \frac{1 - |\alpha|^2}{|z - \alpha|^2}d\theta; \end{aligned}$$

[3] See [**Ahlfors**] for the general formula on an R-disk.

then we obtain

$$u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{\mathbf{i}\theta})\frac{1-|\alpha|^2}{|e^{\mathbf{i}\theta}-\alpha|^2}d\theta \tag{IV.B.1}$$

Again for $z = e^{\mathbf{i}\theta}$, noting that

$$\frac{z+\alpha}{z-\alpha} = \frac{(z+\alpha)(\bar{z}-\bar{\alpha})}{|z-\alpha|^2} = \frac{1-|\alpha|^2+\mathbf{i}(\dots)}{|z-\alpha|^2},$$

this becomes

$$u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{\mathbf{i}\theta})\,\mathrm{Re}\left\{\frac{e^{\mathbf{i}\theta}+\alpha}{e^{\mathbf{i}\theta}-\alpha}\right\}d\theta. \tag{IV.B.2}$$

Finally, writing $\alpha = ae^{\mathbf{i}\psi}$ and

$$\begin{aligned}|e^{\mathbf{i}\theta}-ae^{\mathbf{i}\psi}|^2 &= (e^{\mathbf{i}\theta}-ae^{\mathbf{i}\psi})(e^{-\mathbf{i}\theta}-ae^{-\mathbf{i}\psi})\\ &= 1-a\{e^{\mathbf{i}(\psi-\theta)}+e^{\mathbf{i}(\theta-\psi)}\}+a^2 = 1-2a\cos(\theta-\psi)+a^2,\end{aligned}$$

yields

$$u(ae^{\mathbf{i}\psi}) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{\mathbf{i}\theta})P_a(\theta-\psi)d\theta \tag{IV.B.3}$$

where

$$P_a(\theta-\psi) := \frac{1-a^2}{1-2a\cos(\theta-\psi)+a^2}$$

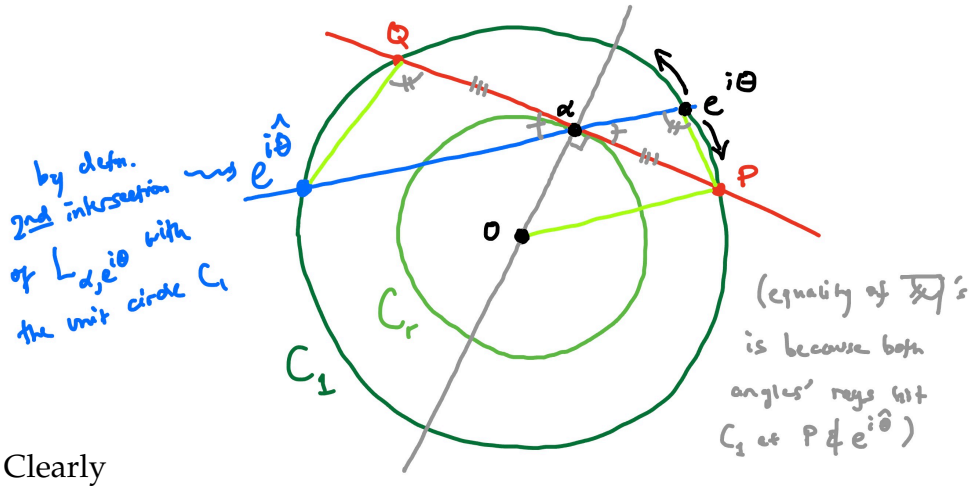
is the **Poisson kernel**. Formulas (IV.B.1)-(IV.B.3) are all versions of the **Poisson Formula**.

The Poisson kernel takes the appealing form

$$\begin{aligned}P_r(\mu) :&= \mathrm{Re}\left(\frac{e^{\mathbf{i}\mu}+r}{e^{\mathbf{i}\mu}-r}\right) = \mathrm{Re}\left(\frac{1+e^{-\mathbf{i}\mu}r}{1-e^{-\mathbf{i}\mu}r}\right) = \mathrm{Re}\left(\frac{1+e^{\mathbf{i}\mu}r}{1-e^{\mathbf{i}\mu}r}\right)\\ &= \mathrm{Re}\{(1+re^{\mathbf{i}\mu})(1+re^{\mathbf{i}\mu}+r^2e^{2\mathbf{i}\mu}+\dots)\}\\ &= \mathrm{Re}\left\{1+2\sum_{n\geq 1} r^n e^{\mathbf{i}n\mu}\right\}\\ &= 1+2\sum_{n\geq 1} r^n\cos(n\mu)\\ &= \sum_{n\in\mathbb{Z}} r^{|n|}e^{\mathbf{i}n\mu}\end{aligned}$$

But what does it *mean*?

Consider the picture[4]



Clearly

- $\Delta P\alpha 0$ right triangle $\implies |\overline{P\alpha}| = |\overline{\alpha Q}| = \sqrt{1-|\alpha|^2}$
- $\Delta \alpha e^{i\theta} P \sim \Delta \alpha Q e^{i\hat\theta} \implies \frac{\sqrt{1-|\alpha|^2}}{|e^{i\hat\theta}-\alpha|} = \frac{|e^{i\theta}-\alpha|}{\sqrt{1-|\alpha|^2}} \implies$

$$1-|\alpha|^2 = |(e^{i\theta}-\alpha)(e^{i\hat\theta}-\alpha)| \tag{IV.B.4}$$

- $\arg(e^{i\theta}-\alpha) \equiv \pi + \arg(e^{i\hat\theta}-\alpha) \implies$

$$\pi = \arg(e^{i\theta}-\alpha) - \arg(e^{i\hat\theta}-\alpha) = \arg(e^{i\theta}-\alpha) + \arg(e^{-i\hat\theta}-\bar\alpha) \implies$$

$$-(e^{i\theta}-\alpha)(e^{-i\hat\theta}-\bar\alpha) = 1-|\alpha|^2 \tag{IV.B.5}$$

using (IV.B.4). Taking $d\log$ of (minus) both sides of (IV.B.5) gives

- $\mathrm{dlog}(e^{i\theta}-\alpha) = -\mathrm{dlog}(e^{-i\hat\theta}-\bar\alpha) \implies \frac{ie^{i\theta}d\theta}{e^{i\theta}-\alpha} = \frac{-ie^{-i\hat\theta}d\hat\theta}{e^{-i\hat\theta}-\bar\alpha}$

$$\implies \frac{d\hat\theta}{d\theta} = -e^{i(\theta+\hat\theta)}\cdot\frac{e^{-i\hat\theta}-\bar\alpha}{e^{i\theta}-\alpha} = \left|\frac{e^{i\hat\theta}-\alpha}{e^{i\theta}-\alpha}\right| \overset{\text{(IV.B.4)}}{\implies} \frac{d\hat\theta}{d\theta} = \frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}.$$

Substituting this into (IV.B.1) gives $u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})d\hat\theta$, hence by symmetry of the correspondence $\theta \leftrightarrow \hat\theta$

$$u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\hat\theta})d\theta, \tag{IV.B.6}$$

the "geometric interpretation" of Poisson's formula.

[4]The equality of edge lengths shown is *not* meant to be an ingredient in the similarity of triangles below.

Next, we show how to weaken the hypotheses a bit.

IV.B.7. THEOREM. *If $u \in \mathcal{C}^0(\overline{D}_1)$ with $u|_{D_1}$ harmonic, then*

$$u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{\mathbf{i}\theta})P_a(\theta-\psi)d\theta$$

for all $\alpha = ae^{\mathbf{i}\psi} \in D_1$.

PROOF. Let $r \in (0,1)$, so that $u \circ \mu_r \in \mathcal{H}(\overline{D}_1)$, where μ_r is multiplication by r. Now Poisson $\implies$

$$u(r\alpha) = (u\circ\mu_r)(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{(u\circ\mu_r)(e^{\mathbf{i}\theta})}_{u(re^{\mathbf{i}\theta})} P_a(\theta-\psi)d\theta.$$

Using (uniform!) continuity of u on $\overline{D}_1$, we get that as $r \to 1^-$, $u(re^{\mathbf{i}\theta}) \to u(e^{\mathbf{i}\theta})$ uniformly in $\|\cdot\|_{[0,2\pi]}$ (and thus the same for its product with the continuous function $P_a(\theta-\psi)$, which doesn't vary). Taking limits on both sides now gives the desired result. □

IV.B.8. COROLLARY. *Under the above hypotheses, the harmonic conjugate is given (up to a constant) by*

$$v(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{\mathbf{i}\theta})\,\mathrm{Im}\left\{\frac{e^{\mathbf{i}\theta}+\alpha}{e^{\mathbf{i}\theta}-\alpha}\right\}d\theta$$

PROOF. Recalling that on ∂D_1 we have $\frac{dz}{z} = \mathrm{dlog}(z) = d(\mathbf{i}\theta) = \mathbf{i}d\theta$, define

$$f(\alpha) := \frac{1}{2\pi}\int_0^{2\pi} u(e^{\mathbf{i}\theta})\cdot\frac{e^{\mathbf{i}\theta}+\alpha}{e^{\mathbf{i}\theta}-\alpha}d\theta = \frac{1}{2\pi\mathbf{i}}\oint_{\partial D_1} u(z)\cdot\frac{z+\alpha}{z-\alpha}\frac{dz}{z},$$

which is in $\mathscr{H}\mathscr{o}\mathscr{l}(D_1)$ (as a function of α), see the Lemma below. Taking real parts gives the Poisson formula (and hence u); so taking imaginary parts gives the harmonic conjugate. □

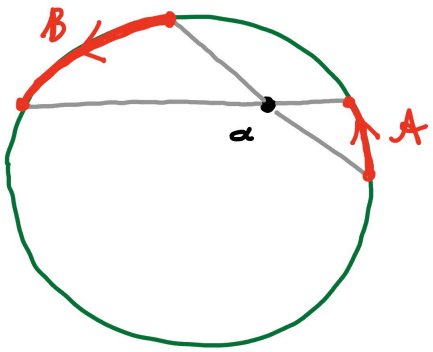
Technically, this uses the following easy application of Morera:

IV.B.9. LEMMA. *Let $\varphi(w,t) \in C^0(U\times[\alpha,\beta])$, with $\varphi(w,t) \in \mathscr{H}\mathscr{o}\mathscr{l}(U)$ for every $t \in [\alpha,\beta]$. Then*

$$F(w) = \int_\alpha^\beta \varphi(w,t)dt \in \mathscr{H}\mathscr{o}\mathscr{l}(U).$$

§ The Dirichlet problem for D_1.

We can also give "physical" interpretations of the various Poisson formulas: in (IV.B.1), we can see the kernel $\frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2}$ (which is itself a harmonic function of α — see the exercises) as the "point-source-at-$e^{i\theta}$ stable temperature distribution." The geometric formula (IV.B.4) interprets $u(\alpha)$ as the (weighted) average of $u|_{\partial D_1}$ with respect to the measure which weights the arc $\mathcal{A}$ by $\frac{1}{2\pi} \times$ (arclength of $\mathcal{B}$):



In particular, since the constant 1 is a harmonic function, we know that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{1-|\alpha|^2}{|e^{i\theta}-\alpha|^2} d\theta = 1. \tag{IV.B.10}$$

We'll now use Poisson's formula to *construct* harmonic functions.

IV.B.11. THEOREM. *Let $f \in \mathcal{C}^0_{\mathbb{R}}(\partial D_1)$, and set*

$$u(z) := \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{1-|z|^2}{|z-e^{i\theta}|^2} d\theta, & z \in D_1 \\ f(z), & z \in \partial D_1 \end{cases}$$

Then $u \in \mathcal{C}^0(\overline{D}_1)$ and $u|_{D_1} \in \mathcal{H}(D_1)$.

PROOF. This proceeds in two steps.

Step 1 [$u|_{D_1}$ is harmonic]

Quick version: $u(z)$ is Re of $\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{e^{i\theta}+z}{e^{i\theta}-z} d\theta$, which is holomorphic by Lemma IV.B.9.

More explicit version: first note that

$$\frac{e^{\mathbf{i}\theta}}{e^{\mathbf{i}\theta}-z}+\frac{e^{-\mathbf{i}\theta}}{e^{-\mathbf{i}\theta}-\bar{z}}-1=$$
$$\frac{(1-e^{-\mathbf{i}\theta}\bar{z})+(1-ze^{-\mathbf{i}\theta})-(1+|z|^2-ze^{-\mathbf{i}\theta}-\bar{z}e^{\mathbf{i}\theta})}{|z-e^{\mathbf{i}\theta}|^2}=\frac{1-|z|^2}{|z-e^{\mathbf{i}\theta}|^2}$$

so (by Poisson) $u(z)=$

$$\frac{1}{2\pi}\int_0^{2\pi}f(e^{\mathbf{i}\theta})\frac{e^{\mathbf{i}\theta}d\theta}{e^{\mathbf{i}\theta}-z}+\frac{1}{2\pi}\int_0^{2\pi}f(e^{\mathbf{i}\theta})\frac{e^{-\mathbf{i}\theta}d\theta}{e^{-\mathbf{i}\theta}-\bar{z}}-\frac{1}{2\pi}\int_0^{2\pi}f(e^{\mathbf{i}\theta})d\theta$$
$$=: \mathrm{I}+\mathrm{II}-\mathrm{III},$$

where I is holomorphic by II.E.12, II is antiholomorphic (apply II.E.12 to its conjugate), and III is constant. Since u is real,

$$u=\operatorname{Re}u=\underbrace{\operatorname{Re}\mathrm{I}+\operatorname{Re}\mathrm{II}-\operatorname{Re}\mathrm{III}}_{\text{all clearly harmonic}}.$$

Step 2 [Continuity at the boundary]

Intuition: the Poisson weighting limits to a δ-function as z moves radially toward ∂D_1. In particular, $\frac{1-|z|^2}{|e^{\mathbf{i}\theta}-z|^2}=0$ for $|z|=1$ and $z\neq e^{\mathbf{i}\theta}$.

Proof: For a piecewise $\mathcal{C}^0$ function $\mathfrak{F}$ on ∂D_1, set

$$P_{\mathfrak{F}}(z):=\frac{1}{2\pi}\int_0^{2\pi}\mathfrak{F}(e^{\mathbf{i}\theta})\frac{1-|z|^2}{|e^{\mathbf{i}\theta}-z|^2}d\theta.$$

By Lemma IV.B.9, if we write $\gamma:=\overline{\{z\in\partial D_1\mid \mathfrak{F}(z)\neq 0\}}$ then $P_{\mathfrak{F}}\in\mathcal{H}(\mathbb{C}\setminus\gamma)$. Fix $e^{\mathbf{i}\theta_0}\in\partial D_1$ and assume $f(e^{\mathbf{i}\theta_0})=0$. (It suffices to consider this case, as we can add a constant to u later.) Choose $\gamma_2:=$ small open arc of ∂D_1 such that $\gamma_2\ni e^{\mathbf{i}\theta_0}$ and $\|f\|_{\gamma_2}<\frac{\epsilon}{2}$. Set $\gamma_1:=\partial D_1\setminus\gamma_2$. Put

$$\mathfrak{F}_i:=\begin{cases}f & \text{on } \gamma_i\\ 0 & \text{on } \gamma_{3-i}\end{cases}$$

Then by IV.B.10

$$|P_{\mathfrak{F}_2}(z)|\leq\frac{\epsilon}{2}\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|z-e^{\mathbf{i}\theta}|^2}d\theta=\frac{\epsilon}{2},\quad\text{for } z\in D_1.$$

Now there exists $\delta > 0$ such that (for $z \in D_1$)

$$|z - e^{\mathbf{i}\theta_0}| < \delta \implies \left\| f(e^{\mathbf{i}\theta}) \frac{1 - |z|^2}{|z - e^{\mathbf{i}\theta}|^2} \right\|_{e^{\mathbf{i}\theta} \in \gamma_1} < \frac{\epsilon}{2} \implies |P_{\mathfrak{F}_1}(z)| < \frac{\epsilon}{2}$$

So $u = P_{\mathfrak{F}_1} + P_{\mathfrak{F}_2}$ (on D_1)

$$\implies |u(z)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \text{for } z \in D(e^{\mathbf{i}\theta_0}, \delta) \cap D_1,$$

which (since $u(e^{\mathbf{i}\theta_0}) = f(e^{\mathbf{i}\theta_0}) = 0$) is the desired continuity statement. □

Exercises

(1) Prove Lemma IV.B.9.

(2) Show that $\frac{R^2 - |z|^2}{|Re^{i\theta_0} - z|^2}$ is harmonic (in z) on D_R.