IV.C. More on harmonic functions

§ Fourier series.

This is a nice application of the solution to Dirichlet's problem on the unit disk, cf. Theorem IV.B.11. Moreover, it gives a better idea of what the harmonic functions of that Theorem look like.

Let $f \in \mathcal{C}^0_{\mathbb{R}}([0,2\pi])$, $f(0) = f(2\pi)$. The Theorem just mentioned guarantees the existence of $u \in \mathcal{C}^0(\overline{D}_1)$ satisfying:

(a) $u(e^{\mathbf{i}\theta}) = f(\theta)$.

(b) $u|_{D_1}$ is harmonic, hence of the form[5]

$$\begin{aligned} u(re^{\mathbf{i}\theta}) &= \mathrm{Re}\left(\sum_{k\geq 0} \alpha_k (re^{\mathbf{i}\theta})^k\right) \\ &= \frac{a_0}{2} + \sum_{k\geq 1} a_k r^k \cos(k\theta) + \sum_{k\geq 1} b_k r^k \sin(k\theta) \end{aligned} \tag{IV.C.1}$$

where the series are absolutely and uniformly convergent on $\overline{D}_r$ for $r < 1$.

Now using basic trigonometric integrals, (IV.C.1) gives

$$\frac{1}{\pi}\int_0^{2\pi} \cos(n\theta) u(re^{\mathbf{i}\theta}) d\theta = a_n r^n,$$
$$\frac{1}{\pi}\int_0^{2\pi} \sin(n\theta) u(re^{\mathbf{i}\theta}) d\theta = b_n r^n.$$

Taking $r \to 1^-$ and using (uniform) continuity of u on $\overline{D}_1$ together with (a),

$$\begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \cos(n\theta) f(\theta) d\theta = a_n \\ \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \sin(n\theta) f(\theta) d\theta = b_n \end{cases} \tag{IV.C.2}$$

This may be regarded as the definition for **Fourier coefficients of f**. Together with (IV.C.1), it gives a *formula for $u|_{D_1}$ as a series*. What about f? This is a nontrivial question, since "the series converges to

[5] Here $a_0 = 2\mathrm{Re}(\alpha_0)$, $a_{k>0} = \mathrm{Re}(\alpha_k)$, and $b_k = -\mathrm{Im}(\alpha_k)$.

u on D_1" and "u is continuous on $\overline{D}_1$" do *not* imply that the series converges to f on ∂D_1.

Since f is (uniformly) continuous, for every $\epsilon > 0$ there must be a $\delta > 0$ for which $|\theta_2 - \theta_1| < \delta \implies |f(\theta_2) - f(\theta_1)| < \frac{\epsilon}{2}$. This means that for $n > \frac{2\pi}{\delta}$ (i.e. sufficiently large),

$$\left| \frac{1}{\pi} \int_{\frac{2\pi}{n}m}^{\frac{2\pi}{n}(m+1)} f(\theta)\cos(n\theta)d\theta \right| = \left| \frac{1}{\pi} \int_{\frac{2\pi}{n}m}^{\frac{2\pi}{n}(m+1)} \left[f(\theta) - f(\tfrac{2\pi}{n}m)\right]\cos(n\theta)d\theta \right| < \frac{1}{\pi}\cdot\frac{2\pi}{n}\cdot\frac{\epsilon}{2} = \frac{\epsilon}{n}$$

since $\int_{\frac{2\pi}{n}m}^{\frac{2\pi}{n}(m+1)} \cos(n\theta)d\theta = 0$. This implies

$$|a_n| = \left| \sum_{m=0}^{n-1} \frac{1}{\pi} \int_{\frac{2\pi}{n}m}^{\frac{2\pi}{n}(m+1)} f(\theta)\cos(n\theta)d\theta \right| < \epsilon, \tag{IV.C.3}$$

whence $a_n \to 0$ as $n \to \infty$ (and similarly for b_n).

But this isn't good enough for convergence: indeed, if $\mathscr{S} \subset [0, 2\pi]$ is *any* set of measure zero, then there exist functions $f \in \mathcal{C}^0([0, 2\pi])$ whose Fourier series diverge (unboundedly!) on $\mathscr{S}$. A famous theorem of Carleson implies that the Fourier series at least converges pointwise almost everywhere, but still this is a bit shocking

(i) when you first learn Fourier series from physicists who repeat the mantra that $f \in \mathcal{C}^\ell([0, 2\pi]) \implies a_n \sim \frac{1}{n^{\ell+2}}$

(ii) in light of our theorem on Dirichlet for D_1. The problem is that, while (according to that theorem) *u limits to $f(e^{\mathbf{i}\theta_0})$ at each point $e^{\mathbf{i}\theta_0} \in \partial D_1$*, this statement amounts to *Abel summability* of the Fourier series at θ_0, which is *weaker* than ordinary summability!

To fix this, suppose now that f is *everywhere differentiable, with bounded derivative* (weaker than $\mathcal{C}^1$). Then $\|f'\|_{[0,2\pi]} \leq M$, and so if $\epsilon = \frac{2\pi M}{n}$ we can take $\delta = \frac{\epsilon}{M} = \frac{2\pi}{n}$ above to get

$$|na_n| < \frac{2\pi M}{\epsilon}\cdot\epsilon = 2\pi M$$

(and the same for b_n) from (IV.C.3).

Under the stronger hypothesis that f is $\mathcal{C}^1$, we can apply the above argument (which produced (IV.C.3)) to f' since it is *uniformly* continuous. Combined with integration by parts, this yields for any $\epsilon > 0$ (and n sufficiently large)

$$|a_n| = \left| \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta \right| = \left| -\frac{1}{n\pi} \int_0^{2\pi} f'(\theta) \sin(n\theta) d\theta \right| < \frac{\epsilon}{n},$$

and we conclude that $|na_n|$ (and $|nb_n|$) $\to 0$ as $n \to \infty$.

In the first case (na_n bounded) we can use Littlewood's theorem,[6] in the second case ($na_n \to 0$) Tauber's theorem (proved in Chapter I.E), to assert that

$$\begin{aligned} f(\theta_0) &= \lim_{r \to 1^-} u(re^{\mathbf{i}\theta_0}) \\ &= \lim_{r \to 1^-} \left(\frac{a_0}{2} + \sum_{n \geq 1} [a_n \cos(n\theta_0) + b_n \sin(n\theta_0)] r^n \right) \\ &= \frac{a_0}{2} + \sum_{n \geq 1} [a_n \cos(n\theta_0) + b_n \sin(n\theta_0)]. \end{aligned}$$

(and that this last expression converges).

Conversely, by Abel's theorem, whenever the last expression converges, the $\lim_{r \to 1^-}$ must equal it; and since the $\lim_{r \to 1^-}$ recovers the boundary value $u(e^{\mathbf{i}\theta_0}) = f(\theta_0)$, we have the

IV.C.4. THEOREM. *Let $f \in \mathcal{C}^0_{\mathbb{R}}([0, 2\pi])$ and a_n, b_n be its Fourier coefficients. Then*

(a) *Whenever $\frac{a_0}{2} + \sum_{n \geq 1} [a_n \cos(n\theta) + b_n \sin(n\theta)]$ (i.e. the **Fourier series of f**) converges, it converges to $f(\theta)$.*

(b) *The Fourier series converges everywhere if f is everywhere differentiable, with bounded derivative.*

[6] We did not prove this theorem, and so part (b) of Theorem IV.C.4 has only been completely proved in the continuously differentiable case.

§ Harmonic differentials.

Though we only want to deal with 1-forms on $\mathbb{C}$, I'll start by putting this in a broader context:

IV.C.5. DEFINITION. If M is an oriented Riemannian n-manifold, one can take an oriented orthonormal basis $\{\theta_i\}$ for local coframes[7] and define the **Hodge star operator** $*$ by

$$\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \wedge *(\theta_{i_1} \wedge \cdots \wedge \theta_{i_k}) := \theta_1 \wedge \cdots \wedge \theta_n.$$

Presently this yields[8]

$$dx \wedge *dx = dx \wedge dy \implies *dx = dy$$
$$dy \wedge *dy = dx \wedge dy \implies *dy = -dx$$

and so $*(A dx + B dy) = -B dx + A dy$. Furthermore

$$*dz = *(dx + \mathbf{i}dy) = -\mathbf{i}dx + dy = -\mathbf{i}(dx + \mathbf{i}dy) = -\mathbf{i}dz$$
$$*d\bar{z} = *(dx - \mathbf{i}dy) = \mathbf{i}dx + dy = \mathbf{i}(dx - \mathbf{i}dy) = \mathbf{i}d\bar{z},$$

which yields $\mathbf{i} * (A dz + B d\bar{z}) = A dz - B d\bar{z}$, and in particular

$$\mathbf{i} * A dz = A dz.$$

Now let ω be a *real* 1-form on a region $U \subset \mathbb{C}$:

$$\omega = a\,dx + b\,dy = A\,dz + B\,d\bar{z}, \quad a, b \text{ real-valued } \mathcal{C}^\infty \text{ functions.}$$

IV.C.6. DEFINITION. ω is a **harmonic differential** if

$$d\omega = 0 = d(*\omega).$$

In this case, $\phi := \omega + \mathbf{i} * \omega = 2A\,dz$ satisfies

$$-2\frac{\partial A}{\partial \bar{z}} dz \wedge d\bar{z} = d\phi = 0 \quad \implies \quad \frac{\partial A}{\partial \bar{z}} = 0 \quad \implies \quad A \in \mathscr{Hol}(U).$$

Since ω real $\implies$ $\omega = \overline{\omega}$ $\implies$ $B = \overline{A}$, we have $\omega = \mathrm{Re}(2A\,dz)$, where $2A\,dz$ is a holomorphic 1-form.

[7] that is, a "basis" (over the ring of C^∞ functions) for differential 1-forms in a neighborhood, which is orthonormal in the metric on the cotangent space at each point.
[8] Notice that $** = -1$ on 1-forms.

Conversely, taking $\phi = 2A\,dz$ with $\frac{\partial A}{\partial \bar{z}} = 0$, write

$$\phi = \frac{1}{2}(\phi + \bar{\phi}) + \mathbf{i} \cdot \frac{1}{2\mathbf{i}}(\phi - \bar{\phi}) =: \omega + \mathbf{i}\eta.$$

Clearly ω and η are *real* 1-forms. We have

$$0 = d\phi = d\omega + \mathbf{i}d\eta \implies d\omega = 0 = d\eta.$$

$$\begin{aligned} \text{But } \mathbf{i} * \phi = \phi &\implies \omega + \mathbf{i}\eta = \mathbf{i}(*\omega + \mathbf{i} * \eta) = - * \eta + \mathbf{i} * \omega \\ &\implies \eta = *\omega \\ &\implies d(*\omega) = 0 \\ &\implies \omega \text{ harmonic.} \end{aligned}$$

We have proved:

IV.C.7. PROPOSITION. *The harmonic differentials ω are precisely the real parts of holomorphic differentials, with $*\omega$ equal to the imaginary part.*

Now again let ω be harmonic. Using Prop. IV.C.7, we have $\omega = \mathrm{Re}(f\,dz)$, with f holomorphic. But f is *locally* the derivative of something else holomorphic, say F. (This holds "globally" in any simply-connected region in which f is defined and holomorphic.) So $\omega = \mathrm{Re}(dF) = d(\mathrm{Re}(F))$ holds locally, and as the real part of a holomorphic function, $\mathrm{Re}(F)$ is harmonic.

Conversely, suppose that u is harmonic and $\omega := du$ (locally). Using $\Delta = *d*d$ (see the exercises), $d(*\omega) = d*du = - * *d*du = - * \Delta u = 0$; and $d\omega = ddu = 0$. So ω is harmonic.

IV.C.8. PROPOSITION. *The harmonic differentials are (locally, or in simply connected regions) precisely the differentials of harmonic functions.*

If u is a harmonic function in a simply connected region then we may define the harmonic conjugate of u as the integral of $*du$, or put differently, $dv = *du$. (That this agrees with the previous definition is an exercise.) This both explains why harmonic conjugates aren't well-defined in general in multiply connected regions, and illustrates why the differential-form point of view is better!

Again let $U \subset \mathbb{C}$ be any region. If $u_1, u_2 \in \mathcal{H}(U)$, then any simply-connected subregion $U_0 \subset U$ has conjugate functions v_1, v_2, with $u_j + \mathbf{i}v_j \in \mathscr{Hol}(U_0)$. Writing $\Omega^1(U_0)$ for the holomorphic 1-forms on U_0, we have $\Omega := (u_1 + \mathbf{i}v_1)\, d(u_2 + \mathbf{i}v_2) \in \Omega^1(U_0)$. Writing

$$\Omega = \mathrm{Re}(\Omega) + \mathbf{i}\{v_1 du_2 + u_1 dv_2\},$$

the homology Cauchy theorem gives $\int_{\partial \mathcal{K}} \Omega = 0$ for any 2-chain $\mathcal{K}$ in U_0, and taking the imaginary part yields

$$0 = \int_{\partial K} v_1 du_2 + u_1 dv_2.$$

Since $u_2 dv_1 + v_1 du_2 = d(u_2 v_1)$ is exact (*a fortiori* closed), we also have $0 = \int_{\partial \mathcal{K}} u_2 dv_1 + v_1 du_2$ and subtracting gives

$$0 = \int_{\partial K} u_1 dv_2 - u_2 dv_1 = \int_{\partial K} u_1 * du_2 - u_2 * du_1.$$

Since a 2-chain on U can be broken into 2-chains on simply-connected subregions, we can add the boundaries up to get

$$0 = \int_{\gamma} u_1 * du_2 - u_2 * du_1 \tag{IV.C.9}$$

for any closed curve γ which is homologous to zero in U. Notice that (IV.C.9) is written in terms of u_i and $*du_i$, which are well-defined on U, as opposed to v_i, which may not be.

An application. We can use harmonic differentials to clarify the behavior of harmonic functions on a *punctured* disk $U = D_R *$. (Writing r, θ for polar coordinates, we will use $*dr = r d\theta$ and $*d\theta = -\frac{1}{r} dr$, which are left as an exercise.)

If $u \in \mathcal{H}(D_R^*)$, then taking $\gamma = C_{r_2} - C_{r_1}$ $(0 < r_1, r_2 < R)$, we want to apply (IV.C.9) to $u_1 = u$ and $u_2 = \log(r)$. First we calculate

$$*du_1 = * \left(\frac{\partial u}{\partial r} dr + \frac{\partial u}{\partial \theta} d\theta \right) = r \frac{\partial u}{\partial r} d\theta - \frac{1}{r} \frac{\partial u}{\partial \theta} dr$$

$$*du_2 = \frac{1}{r} r d\theta = d\theta$$

and also set $u_3 = 1 \implies *du_3 = 0$. Now (IV.C.9) gives

$$0 = \int_\gamma u_1 * du_2 - u_2 * du_1 = \int_\gamma u d\theta - \int_\gamma \log(r) * du$$

and

$$0 = \int_\gamma u_1 * du_3 - u_3 * du_1 = -\int_\gamma *du.$$

So

$$\int_{C_r} *du =: 2\pi\alpha$$

and

$$\int_{C_r} u d\theta \; - \; \log(r) \int_{C_r} *du \quad (=: 2\pi\beta)$$

are both constant in $r \in (0, R)$, and we conclude that

$$\frac{1}{2\pi} \int_{C_r} u d\theta = \beta + \alpha \log(r) \quad (\forall r \in (0, R)). \tag{IV.C.10}$$

This generalizes the Mean Value Theorem, which is the case $\alpha = 0$ where $u \in \mathcal{H}(D_R)$ with $u(0) = \beta$.

Exercises

(1) Show that the Laplacian $\Delta = *d * d$.

(2) Hodge star in polar coordinates: check that $*dr = r\, d\theta$ and $*d\theta = -\frac{1}{r} dr$.

(3) If U is simply connected, $u \in \mathcal{H}(U)$, and $p \in U$, show that $v(z) = \int_p^z *du$ defines a harmonic conjugate of u in the sense of IV.A.8.

(4) (i) If $f(w) \in \mathscr{H}\mathscr{o}\mathscr{l}(D_r^*)$ with $\lim_{w\to 0} w \cdot Re(f(w)) = 0$, show that 0 is a removable singularity of f. [Hint: convert to $z = \frac{1}{w}$ and look at problem 7 on p. 171 of [**Ahlfors**].]

(ii) If $u(z) \in \mathcal{H}(D_\rho^*)$ with $\lim_{z\to 0} z \cdot u(z) = 0$, show that $u(z) = u_0(z) + \alpha \log|z|$ for $\alpha \in \mathbb{C}$ and $u_0 \in \mathcal{H}(D_\rho)$. [Hint: use the result of part (i) and the setup of the application above. Keep in mind the meaning of $\int *du$.]

(5) (i) Using Theorem IV.C.4(a) on Fourier series, solve the Dirichlet problem in the disk for the functions on $[-\pi, \pi]$ (i.e. on ∂D_1)

(a) $f(\theta) = \begin{cases} 1, & |\theta| \le \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < |\theta| \le \pi \end{cases}$

(b) $f(\theta) = 1 - \frac{2}{\pi}|\theta|$
(c) $f(\theta) = (|\theta| - \pi)^2 - \frac{\pi^2}{3}$
First, express your answer as a series, then try to identify the holomorphic function of which u is the real part.
(ii) Using your answer to part (i), evaluate
(a) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$
(b) $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$
(c) $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots$