IV.D. Extensions and boundary values

§ Schwarz reflection.

Let $U \subseteq \mathbb{C}$ be a region and write

$$U^* := \{z \mid \bar{z} \in U\}$$

for its reflection about the real axis. (I don't write $\overline{U}$ because that is already used for closure.)

IV.D.1. LEMMA. (i) *If $f(z) \in \mathscr{Hol}(U)$, then $\overline{f(\bar{z})} \in \mathscr{Hol}(U^*)$.*
(ii) *If $u \in \mathcal{H}(U)$, then $u(\bar{z}) \in \mathcal{H}(U^*)$.*

PROOF. (i): There are a few ways to see this: e.g.,

- $\frac{\partial}{\partial \bar{z}}\overline{f(\bar{z})} = \overline{\frac{\partial}{\partial z} f(\bar{z})} = \overline{\frac{\partial f}{\partial \bar{z}}\frac{\partial z}{\partial z} + \frac{\partial f}{\partial z}\frac{\partial \bar{z}}{\partial z}} = \overline{0\frac{\partial z}{\partial z} + \frac{\partial f}{\partial z}0} = 0$
- locally $f(z) = \sum \alpha_n z^n \implies f(\bar{z}) = \sum \alpha_n \bar{z}^n \implies \overline{f(\bar{z})} = \sum \bar{\alpha}_n z^n$.

(ii): In a neighborhood of any point $z_0 \in U$, $u = \mathrm{Re}(f_0)$ with f_0 holomorphic. Then $f_0(z) = u(z) + \mathbf{i}v_0(z) \implies \overline{f_0(\bar{z})} = u(\bar{z}) - \mathbf{i}v_0(\bar{z})$ holomorphic $\implies u(\bar{z})$ harmonic. □

Now we will assume henceforth that

$$U = U^*,$$

i.e. U is symmetric about the real axis. Since U is open it must contain a point in $\mathfrak{H} \cup \mathfrak{H}^*$, say $z_0 \in \mathfrak{H}$, hence also $\bar{z}_0 \in \mathfrak{H}^*$. Since it is connected, it contains a path from z_0 to $\bar{z}_0$, hence (by the intermediate value theorem!) a point of $\mathbb{R}$. Again using openness, U contains an interval $(a, b) \subset \mathbb{R}$. Write

$$\begin{aligned} U \cap \mathfrak{H} &=: U^+ \\ U \cap \mathbb{R} &=: \mathcal{S} \\ U \cap \mathfrak{H}^* &=: U^- (= (U^+)^*). \end{aligned}$$

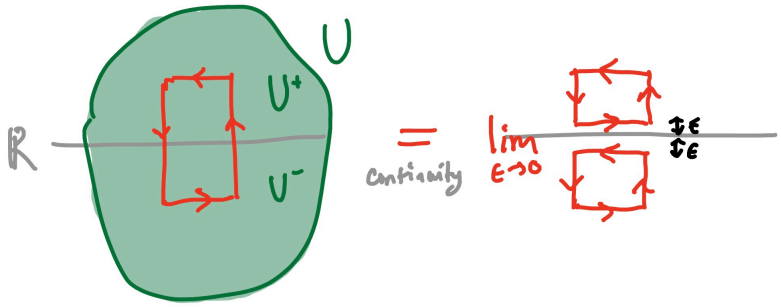
The rough idea of Schwarz reflection is this:

IV.D.2. PROPOSITION. *Let $f \in \mathcal{C}^0(U^+ \cup \mathcal{S})$ with $f|_{U^+}$ holomorphic, and $f|_{\mathcal{S}}$ real. Then there exists $g \in \mathscr{Hol}(U)$ with $g|_{U^+ \cup \mathcal{S}} = f$.*

PROOF. Define

$$g(z) := \begin{cases} \overline{f(\bar{z})}, & z \in U^- \\ f(z), & z \in U^+ \cup \mathcal{S}. \end{cases}$$

Since $\overline{f(\bar{z})} = f(z)$ on $\mathcal{S}$, $g \in \mathcal{C}^0(U)$. Clearly $g|_{U^+ \cup U^-}$ is holomorphic (use the Lemma). We need to check that g is holomorphic on $\mathcal{S}$. To do this, apply Morera to its integrals over rectangles



which vanish due to holomorphicity of g on $U^+ \cup U^-$. □

The problem is that this isn't a very strong result: we don't want to assume that f extends continuously to $(U^+ \cup) \mathcal{S}$, just that $\mathrm{Im}(f)$ extends continuously to $\mathcal{S}$ with the value zero there. To do this we first need to prove a reflection principle for harmonic functions:

IV.D.3. THEOREM. *Let $v \in \mathcal{C}^0(U^+ \cup \mathcal{S})$, with $v|_{U^+}$ harmonic and $v|_{\mathcal{S}} \equiv 0$. Then v extends to a harmonic function on U with $v(\bar{z}) = -v(z)$.*

PROOF. Set

$$V(z) := \begin{cases} v(z), & z \in U^+ \\ 0, & z \in \mathcal{S} \\ -v(\bar{z}), & z \in U^-. \end{cases}$$

By hypothesis this belongs to $C^0(U) \cap \mathcal{H}(U^+ \cup U^-)$. We need to check "harmonicity on $\mathcal{S}$." Let $x_0 \in \mathcal{S}$, $D := D(x_0, \epsilon) \subset U$, and put (for all $\alpha \in D$)

$$P_V(\alpha) := \frac{1}{2\pi} \int_{\partial D} V(x_0 + \epsilon e^{i\theta}) \mathrm{Re}\left(\frac{\epsilon e^{i\theta} + (\alpha - x_0)}{\epsilon e^{i\theta} - (\alpha - x_0)} \right) d\theta.$$

By the solution to Dirichlet in D, this extends continuously to V on the boundary ∂D (since V is continuous). Moreover,

$$P_V \in \mathcal{H}(D) \quad \Longrightarrow \quad V - P_V \in \mathcal{H}(D \cap U^+)$$

and

$$(V - P_V)|_{\partial D \cap U^+} \equiv 0.$$

Also, if $\alpha \in \mathcal{S} \cap D$ then (writing $\alpha_0 = \alpha - x_0$)

$$\begin{aligned} P_V(\alpha) &= \frac{1}{2\pi}\int_0^{\pi} v(x_0 + \epsilon e^{\mathbf{i}\theta}) \mathrm{Re}\left(\frac{\epsilon e^{\mathbf{i}\theta} + \alpha_0}{\epsilon e^{\mathbf{i}\theta} - \alpha_0}\right) d\theta \qquad [v(z) \text{ part}] \\ &\quad - \frac{1}{2\pi}\int_{\pi}^{2\pi} v(x_0 + \epsilon e^{-\mathbf{i}\theta'}) \mathrm{Re}\left(\frac{\epsilon e^{\mathbf{i}\theta'} + \alpha_0}{\epsilon e^{\mathbf{i}\theta'} - \alpha_0}\right) d\theta' \quad [-v(\bar{z}) \text{ part}] \\ &= \frac{1}{2\pi}\int_0^{\pi} v(x_0 + \epsilon e^{\mathbf{i}\theta}) \mathrm{Re}\left(\frac{\epsilon e^{\mathbf{i}\theta} + \alpha_0}{\epsilon e^{\mathbf{i}\theta} - \alpha_0}\right) d\theta \\ &\quad + \frac{1}{2\pi}\int_{\pi}^{0} v(x_0 + \epsilon e^{\mathbf{i}\theta}) \mathrm{Re}\left(\frac{\epsilon e^{-\mathbf{i}\theta} + \alpha_0}{\epsilon e^{-\mathbf{i}\theta} - \alpha_0}\right) d\theta \quad [\text{use } \theta = 2\pi - \theta'] \\ &= 0, \end{aligned}$$

since $\mathrm{Re}\left(\frac{\epsilon e^{-\mathbf{i}\theta} + \alpha_0}{\epsilon e^{-\mathbf{i}\theta} - \alpha_0}\right) = \mathrm{Re}\left(\frac{\epsilon e^{\mathbf{i}\theta} + \alpha_0}{\epsilon e^{\mathbf{i}\theta} - \alpha_0}\right)$ for $\alpha_0 \in \mathbb{R}$. So $P_V|_{\mathcal{S}\cap D} \equiv 0$, while meanwhile (by definition) $V|_{\mathcal{S}} \equiv 0$

$$\Longrightarrow \quad (V - P_V)|_{\partial(D \cap U^+)} \equiv 0$$

using $\partial(D \cap U^+) = (\partial D \cap U^+) \cup (\mathcal{S} \cap D)$. So we have that $V - P_V$ is harmonic on $D \cap U^+$, continuous on $\overline{D \cap U^+}$, and $\equiv 0$ on the boundary. By the maximum and minimum principles for harmonic functions, this yields

$$V - P_V \equiv 0 \text{ on } D \cap U^+.$$

Repeating the argument for $D \cap U^-$ we finally have

$$V = P_V \text{ on } D.$$

Since P_V was harmonic there, so is V. We have established harmonicity in a neighborhood of each point of $\mathcal{S}$, and so we are done. $\square$

IV.D.4. EXAMPLES. Writing $z = x + \mathbf{i}y$, $v(z) = xy$ and $e^x \sin(y)$ are harmonic functions which are 0 on $\mathbb{R}$.

Now we are ready for the holomorphic analogue.

IV.D.5. THEOREM. *Let $F \in \mathscr{Hol}(U^+)$ satisfy*

$$\lim_{U^+ \ni z \to x} \mathrm{Im}(F(z)) = 0 \quad (\forall x \in \mathcal{S}).$$

Then

$$\lim_{U^+ \ni z \to x} \mathrm{Re}(F(z)) =: \phi(x)$$

exists for all $x \in \mathcal{S}$, and

$$G(z) := \begin{cases} F(z), & z \in U^+ \\ \phi(z), & z \in \mathcal{S} \\ \overline{F(\bar{z})}, & z \in U^- \end{cases}$$

extends F to a holomorphic function on U.

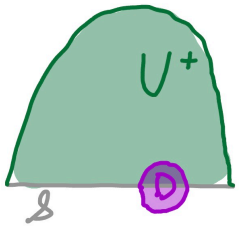
PROOF. Let $x_0 \in \mathcal{S}$, $D := D(x_0, \epsilon) \subset U$. Set

$$v := \mathrm{Im}(F) \in \mathcal{H}(D \cap U^+).$$

Clearly v satisfies the hypotheses of Theorem IV.D.3, so extends to $v \in \mathcal{H}(D)$ satisfying $v(\bar{z}) = -v(z)$. We choose a harmonic conjugate $-u := \int *dv$ on D, making $G_0(z) = u(z) + \mathbf{i}v(z) \in \mathscr{Hol}(D)$, and compute

$$\mathrm{Im}(F - G_0) = \mathrm{Im}(F) - v \equiv 0 \qquad \text{on } D \cap U^+$$

$$\implies F - G_0 \text{ is a real constant there.}$$



Changing u by a (real) constant, we get that $G_0 = F$ on $D \cap U^+$. So $\phi(x)$ exists ($= u(x)$) for $x \in D \cap \mathcal{S}$, and G_0 extends F to D. By Lemma IV.D.1, we have $\overline{G_0(\bar{z})} \in \mathscr{Hol}(D)$ also. Since $v \equiv 0$ on $D \cap \mathcal{S}$, G_0 is real there $\implies (G_0(z) - \overline{G_0(\bar{z})})|_{D \cap \mathcal{S}} \equiv 0 \implies G_0(z) = \overline{G_0(\bar{z})}$ on D. We conclude that $G_0 = G|_D$ so that G is holomorphic on D, which is all that was in question. □

IV.D.6. COROLLARY. *Suppose $f \in \mathcal{C}^0(\overline{\mathfrak{H}})$ with $f|_{\mathfrak{H}}$ holomorphic and $f|_{(a,b)} = 0$. Then $f \equiv 0$ on $\mathfrak{H}$.*

PROOF. Let $x_0 \in (a,b)$ and $(x_0 - \epsilon, x_0 + \epsilon) \subset (a,b)$; restrict f to $D(x_0,\epsilon) \cap \overline{\mathfrak{H}}$. By Theorem IV.D.5, we get a holomorphic extension g to $D(x_0,\epsilon)$ which has $g|_{(x_0-\epsilon,x_0+\epsilon)} \equiv 0 \implies g \equiv 0 \implies f|_{D\cap\mathfrak{H}} \equiv 0 \implies f \equiv 0$. □

§ Some remarks on the Dirchlet problem.

We begin with an application of the result on harmonic functions on a punctured disk from the end of the last Chapter.

IV.D.7. PROPOSITION. *If $u \in \mathcal{H}(D_\rho^*)$ is bounded, then u extends to a harmonic function on D_ρ.*

PROOF. We had that for $0 < r < \rho$,

$$\int_{C_r} u\, d\theta = \beta + \left(\int_{C_r} *du\right) \log r,$$

with $\int_{C_r} *du$ constant in r. Since u is bounded, the left-hand side is bounded as $r \to 0^+$, and so $\int_{C_r} *du$ must equal 0. Hence u has a well-defined harmonic conjugate $\int *du$ on D_ρ^* and is thus the real part of a global holomorphic function f (on D_ρ^*). Expanding f in Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^n = \sum_{n\in\mathbb{Z}} (a_n + \mathbf{i}b_n) z^n$$
$$\implies u(z) = u(re^{\mathbf{i}\theta}) = \mathrm{Re}(f) = \sum_{n\in\mathbb{Z}} \left(a_n \cos(n\theta) - b_n \sin(n\theta)\right) r^n$$

makes it clear that f can't have a pole — this would yield $u(re^{\mathbf{i}\theta}) = \frac{1}{r^m}(\sum_{\ell\geq 0} P_\ell(\theta) r^\ell)$ — while if f had an essential singularity then on any D_ϵ^* it would come close to arbitrary complex values, meaning in particular that u would take arbitrarily large values there. So f must have $\alpha_n = 0$ for every $n < 0$, and the result follows. □

IV.D.8. EXAMPLE. ("Solve the Dirichlet problem for $U = D_1^*$, where the boundary value $f \in C^0_{\mathbb{R}}(\partial U)$ is given, with $f(0) = 0$.") [Note that $\partial U = \{0\} \sqcup \partial D_1$.]

Suppose u is a solution: $u \in C^0(\overline{D}_1)$ with $u(0) = 0$, $u|_{\partial D_1}$ agreeing with f, $u|_{D_1^*}$ harmonic. In particular, u is bounded in D_1^* and (by Prop. IV.D.7) extends to a harmonic function on D_1 (which, by continuity, must satisfy $u(0) = 0$). Thus by the Mean Value Theorem,

$$0 = u(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{\mathbf{i}\theta}) d\theta,$$

so the problem is only solvable if the integral of f around the boundary is zero. This condition is also sufficient, by our solution (using the Poisson kernel) to the Dirichlet problem on D_1.

Assuming the Riemann Mapping Theorem, we should always be able to solve Dirichlet on a simply connected region $U \subsetneq \mathbb{C}$ *provided* $\overline{U}$ *is compact*. The latter condition is unpleasant, and to get around it we need to first extend Dirichlet on D_1 to the setting where f is *piecewise* continuous and *bounded* on ∂D_1. There is in fact nothing in the proof of Theorem IV.B.11 that used continuity on γ_1 (just on γ_2), so we have:

IV.D.9. THEOREM. *Let f be a real, bounded, piecewise continuous function on ∂D_1, discontinuous at Σ, and define $u(z)$ by the Poisson formula as in IV.B.11. Then $u \in \mathcal{C}^0(\overline{D}_1 \setminus \Sigma)$, and $u|_{D_1} \in \mathcal{H}(D_1)$.*

IV.D.10. REMARK. Technically, we need a version of Lemma IV.B.9 for *piecewise* continuous functions, which is obtained by working on the intervals where f is continuous *separately* and then summing the results.

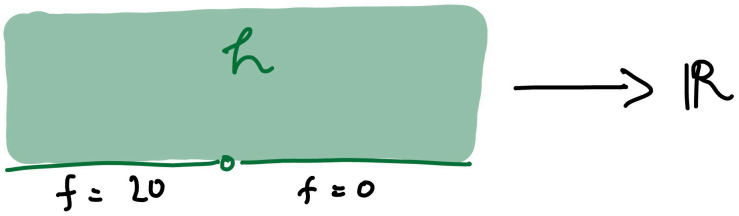
Assuming RMT plus a result on extension of the mapping to the boundary, we get the:

IV.D.11. COROLLARY. *Let $U \subsetneq \mathbb{C}$ be a simply connected region, with ∂U locally connected, and $f\colon \partial U \to \mathbb{R}$ bounded and piecewise continuous. Then there exists $u \in \mathcal{H}(U)$ extending continuously to the boundary minus points of discontinuity $\mathscr{D}$, with value f on $\partial U \setminus \mathscr{D}$.*

SKETCH. The RMT provides an analytic isomorphism $F\colon D_1 \xrightarrow{\cong} U$ extending to a continuous map $\tilde{F}\colon \overline{D}_1 \to \widehat{U}$, where $\widehat{U}$ is the closure in $\hat{\mathbb{C}}$. Let Σ be the preimage of $\mathscr{D}$ and ∞ (if $\infty \in \widehat{U}$). Composing $\partial\overline{D}_1 \setminus \Sigma \xrightarrow{\tilde{F}} \partial U \setminus \mathscr{D} \xrightarrow{f} \mathbb{R}$ and applying the Poisson formula gives $u\colon D_1 \to \mathbb{R}$ harmonic, which (precomposing with F^{-1}) yields the solution on U. □

The main reason for bringing this up before RMT is so we can try to solve Dirichlet in some easy cases:

IV.D.12. EXAMPLE. Does there exist a harmonic function on the upper half plane



with the boundary values displayed on $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$?

Don't even bother with the disk: $\log(z) = \log|z| + \mathbf{i}\arg(z)$ is a well-defined holomorphic function on $\mathfrak{H}$; thus $\log|z|, \arg(z)$ belong to $\mathcal{H}(\mathfrak{H})$. Moreover, $\arg(z)$ is just θ, which is 0 where $f = 0$ and π where $f = 20$. So $\frac{20}{\pi}\arg(z) =: u(z)$ does it.

In fact, this example on the upper half-plane is useful for solving problems elsewhere by explicitly constructing conformal maps to $\mathfrak{H}$.

Exercises

(1) Using the method just suggested, find a harmonic function on
   (a) the unit disk which has boundary values 0 on the lower semicircle and 1 on the upper semicircle;
   (b) the upper unit semidisk $D_1 \cap \mathfrak{H}$ which has boundary values 0 on the semicircle and $[-1, 0]$ and 1 on $[0, 1]$;
   (c) the first quadrant which has boundary values 0 on $[0, 1]$ and 1 on $[1, \infty]$ and $[0, i\infty]$.

(2) If f is analytic in the whole plane and real on the real axis, purely imaginary on the imaginary axis, show that $f(z)$ is odd. [Hint: Schwarz reflection]

(3) Let f be a function in $C^0(\overline{D_1})$ which is holomorphic on D_1, with $|f| = 1$ on ∂D_1. Show that f is (the restriction to $\overline{D_1}$ of) a rational function. [Hint: Schwarz reflection in ∂D_1 is usually given by $\overline{f(1/\bar{z})}$. Try $1/\overline{f(1/\bar{z})}$ instead.]