V. Infinite products

V.A. Functions with prescribed principal parts

The last part of this course is about infinite products, but as a warm-up we shall begin by revisiting infinite sums. The technique we will use for getting these particular function series to converge motivates the construction of the factors in a Weierstrass product.

§ Constructing meromorphic functions on $\mathbb{C}$.

Let U be a region in $\mathbb{C}$. We begin by recalling from III.E.10 the estimate on the Taylor remainder for $f \in \mathscr{Hol}(U)$ at $a \in U$. First, for each $n \in \mathbb{Z}_{\geq 0}$ we can define a function $f_{n+1} \in \mathscr{Hol}(U)$ by

$$f(z) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(z-a)^k + f_{n+1}(z)(z-a)^{n+1}. \tag{V.A.1}$$

Writing $\overline{D} := \overline{D}(a,r) \subset U$, for $z \in D$ we have

$$f_{n+1}(z) = \frac{1}{2\pi\mathbf{i}} \int_{\partial D} \frac{f_{n+1}(\xi)d\xi}{\xi - z}$$

$$= \frac{1}{2\pi\mathbf{i}} \int_{\partial D} \frac{f(\xi)d\xi}{(\xi-a)^{n+1}(\xi-z)} - \frac{1}{2\pi\mathbf{i}} \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} \int_{\partial D} \frac{d\xi}{(\xi-a)^{n-k+1}(\xi-z)}.$$

The $k = n$ term at the end is (omitting a constant)

$$\frac{1}{z-a} \int_{\partial D} \left(\frac{1}{\xi - z} - \frac{1}{\xi - a} \right) d\xi = 0.$$

The other terms in the summation are (up to constants) derivatives of this in a and are therefore zero as well. So

$$|f_{n+1}(z)| \leq \frac{M}{r^n(r - |z-a|)} \quad \text{where } M = \|f\|_{\partial D}$$

and the estimate on the remainder term $f_{n+1}(z)(z-a)^{n+1}$ in (V.A.1) is

$$|f_{n+1}(z)||z-a|^{n+1} \leq \frac{M|z-a|^{n+1}}{r^n(r-|z-a|)}. \tag{V.A.2}$$

In the special case where z is restricted to $\overline{D}_0 := \overline{D}(a, \frac{r}{2}) \subset D$, with a slight weakening and assuming $a = 0$ this becomes

$$|f_{n+1}(z)||z|^{n+1} \leq \frac{M|z|^{n+1}}{(\frac{r}{2})^n(\frac{r}{2})} = M\left(\frac{2|z|}{r}\right)^{n+1}. \tag{V.A.3}$$

We want to build meromorphic functions from *principal parts*.

V.A.4. DEFINITION. For $F \in \mathscr{M}er(U)$ with a pole at α, and Laurent expansion about α

$$F(z) = \sum_{-m \leq n < 0} A_n(z-\alpha)^n + \text{holomorphic},$$

$P(z) := \sum_{-m \leq n<0} A_n(z-\alpha)^n$ is called the **principal part of F at a**.

So the idea will be to add a holomorphic function on U to a sum of principal parts at distinct points $\{b_\nu\} \subset U$. When this set is infinite, the convergence question is interesting. It is also clear, by the definition of a meromorphic function (poles must be isolated in U) that *the $\{b_\nu\}$ may not have an accumulation point **in U***; but they will have at least one on $\partial U \cup \{\infty\}$ if they are infinite in number. In what follows we take $U = \mathbb{C}$.

Let $\{b_\nu\}_{\nu \geq 1} \subset \mathbb{C}$ be a sequence with $|b_\nu| \to \infty$, and $\{P_\nu\}$ a sequence of polynomials with zero constant term; we may assume $b_\nu \neq 0$ $(\forall \nu)$ without loss of generality. The goal is to sum

$$\sum_{\nu \geq 1} P_\nu\left(\frac{1}{z-b_\nu}\right),$$

or (failing that)

$$\sum_{\nu \geq 1} \left\{ P_\nu\left(\frac{1}{z-b_\nu}\right) - \text{holo}_\nu \right\}$$

which will at least have the same principal parts.

Writing $D_r = D(0,r)$, set

$$M_\nu := \left\| P_\nu\left(\frac{1}{z-b_\nu}\right)\right\|_{D_{|b_\nu|/2}}$$

and (for now) let n_ν be an arbitrary increasing sequence of natural numbers. Expanding $P_\nu\left(\frac{1}{z-b_\nu}\right)$ about the origin in $D_{\frac{|b_\nu|}{2}+\epsilon}$, write $\mathfrak{p}_\nu(z)$ for the n_ν-th partial sum. Using $r = |b_\nu|/2$ in (V.A.3), the remainder is bounded by

$$\left|P_\nu\left(\frac{1}{z-b_\nu}\right) - \mathfrak{p}_\nu(z)\right| \le M_\nu\left(\frac{4|z|}{|b_\nu|}\right)^{n_\nu+1} \qquad \forall z \in \overline{D}_{|b_\nu|/4}.$$

That is, the series we want to sum is dominated by the power series

$$\sum_\nu \frac{M_\nu 4^{n_\nu+1}}{|b_\nu|^{n_\nu+1}} z^{n_\nu+1} =: z\sum_\nu a_{n_\nu} z^{n_\nu},$$

which converges absolutely and uniformly on $\overline{D}_R$ if and only if $\limsup_{\nu\to\infty} \sqrt[n_\nu]{|a_{n_\nu}|} < \frac{1}{R}$. In particular, if

$$\lim_{\nu\to\infty} \frac{M_\nu^{1/n_\nu}}{|b_\nu|} = 0, \tag{V.A.5}$$

then taking $N(R)$ such that $\nu \ge N(R) \implies |b_\nu| > 4R$, we have that

$$\sum_{\nu\ge N(R)} \left(P_\nu\left(\frac{1}{z-b_\nu}\right) - \mathfrak{p}_\nu(z)\right) \textit{ converges uniformly on } \overline{D}_R.$$

Consequently the tails of the series converge to holomorphic functions on arbitrarily large balls, and so **the entire series converges to a meromorphic function on $\mathbb{C}$ with the desired principal parts**.

To arrange condition (V.A.5), we simply demand that

$$n_\nu > \log M_\nu,$$

since then

$$\log\frac{M_\nu^{1/n_\nu}}{|b_\nu|} = \underbrace{\frac{\log M_\nu}{n_\nu}}_{<1} - \log|b_\nu| \to -\infty.$$

(Note that since the condition that $n_{\nu+1} > n_\nu$ is already in place, we are really assuming something more like $n_\nu > \nu + \log M_\nu$.)

§ Runge's theorem.

On a disk, an arbitrary holomorphic function is the limit of a sequence of polynomials. This does *not* work on an arbitrary region: already $\frac{1}{z}$ on D_1^* furnishes a counterexample, since there is no polynomial with $\oint_{|z|=1/2} P(z)dz = 2\pi\mathbf{i}$ (or even nonzero). But if we replace "polynomial" by "rational function," we get a nice result:

V.A.6. THEOREM. *Given $K \subset U \subset \mathbb{C}$ (with K compact, U open), $E \subseteq \hat{\mathbb{C}} \setminus K$ a subset with points in every connected component, and $f \in \mathscr{Hol}(U)$. Then for any $\epsilon > 0$ there exists a rational function $R(z)$ with polar set contained in E and $|f(z) - R(z)| < \epsilon$ on K.*

With this in hand, we'll be able to prove a more general result on "constructing meromorphic functions from principal parts." We begin with the following easy

V.A.7. LEMMA. *Given a path $\gamma : [0,1] \to \mathbb{C} \setminus K$, a function $f \in C^0(\gamma)$, and*

$$F(z) := \int_\gamma \frac{f(w)}{w-z} dw.$$

Then for any $\epsilon > 0$, there exists a rational function $R(z)$ with polar set contained in γ, and $|F(z) - R(z)| < \epsilon$ on K.

PROOF. Let $r \in (0, d(\gamma, K))$, $c \geq \max\{\|z\|_K, \|\gamma\|_{[0,1]}, \|f\|_\gamma\}$, and $\alpha, \beta \in \gamma$. Then for $z \in K$,

$$\left|\frac{f(\alpha)}{\alpha - z} - \frac{f(\beta)}{\beta - z}\right| \leq \frac{1}{r^2} |\underbrace{\beta f(\alpha) - \alpha f(\beta) - z\{f(\alpha) - f(\beta)\}}_{= f(\alpha)(\beta-\alpha) + (\alpha - z)(f(\alpha) - f(\beta))}|$$

$$\leq \frac{c}{r^2}|\beta - \alpha| + \frac{2c}{r^2}|f(\alpha) - f(\beta)|.$$

By (uniform) continuity of γ and f, there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that

$$\left|\frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1}) - z}\right| < \frac{\epsilon}{L(\gamma)} \quad \forall t \in [t_{j-1}, t_j].$$

Setting

$$R(z) := \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))(\gamma(t_j) - \gamma(t_{j-1}))}{\gamma(t_{j-1}) - z},$$

we have

$$\begin{aligned} |F(z) - R(z)| &= \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left\{ \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1}) - z} \right\} \gamma'(t) dt \right| \\ &< \frac{\epsilon}{L(\gamma)} \cdot L(\gamma) = \epsilon. \end{aligned}$$

□

Define a distance function ρ on $\mathcal{C}^0(K)$ by

$$\rho(f, g) = \|f - g\|_K.$$

Since the uniform limit of a sequence of $\mathcal{C}^0$ functions on K is C^0, this makes $\mathcal{C}^0(K)$ into a complete metric space. Define

$$B(E) := \overline{\{\text{rational functions with poles } \subset E\}},$$

where the bar denotes closure in $\mathcal{C}^0(K)$ with respect to ρ, i.e. uniform limits of the restrictions of such rational functions to K. $B(E)$ is obviously closed under sums, products, and scalar multiples.

V.A.8. LEMMA. *$\frac{1}{z-a} \in B(E)$ for every $a \notin K$.*

PROOF. Assume first that $\infty \notin E$; set $V := \{a \mid \frac{1}{z-a} \in B(E)\}$. Clearly $E \subset V \subset \mathbb{C} \setminus K$.

Claim: If $a \in V$ and $|b - a| < d(a, K)$, then $b \in V$ ($\implies$ V open). [Proof: $B(E) \ni \frac{1}{z-a} \implies B(E) \ni \frac{1}{z-a} \sum_{n=0}^{N} \left(\frac{b-a}{z-a}\right)^n \implies B(E) \ni \frac{1}{z-a} \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right)^n = \frac{1}{z-b}$ by uniform convergence on K.]

We can use this to show that V is closed in $\mathbb{C} \setminus K$: given $b \in \partial V$, let $V \ni a_n \to b$. Since V is open, $b \notin V$; and applying the contrapositive of the Claim $\implies$ $|b - a_n| \geq d(a_n, K)$. Taking $n \to \infty$ yields $0 = d(b, K)$, and so $b \in K$ (as K compact). Conclude that $\partial V \subset K$.

Let H be any component of $\mathbb{C} \setminus K$. By assumption $H \cap E \neq \emptyset$, so $H \cap V \neq \emptyset$. But $\partial V \subset K \implies H \cap \partial V = \emptyset$. If H has points both not in V and in V, path connectedness shows $H \cap \partial V \neq \emptyset$, a

contradiction; so in fact $H \subset V$. Since H was arbitrary, $\mathbb{C}\backslash K \subset V$ and so $V = \mathbb{C}\backslash K$, and we're done in this case.

Next suppose $\infty \in E$, and set $E_0 := E\backslash\{\infty\} \cup \{a_0\}$ for $a_0 \in (\mathbb{C} \setminus K)^{\text{ub}}$ with $|a_0| > 2\|z\|_K$. (This is in $\mathbb{C}$ and still meets each component of $\hat{\mathbb{C}} \setminus K$.) By the above, $B(E_0) \ni \frac{1}{z-\alpha}$ $(\forall \alpha \in \mathbb{C} \setminus K)$. Moreover, since $\|\frac{z}{a_0}\|_K \leq \frac{1}{2}$,

$$\frac{1}{z-a_0} = -\frac{1}{a_0} \sum_{n=0}^{\infty} \left(\frac{z}{a_0}\right)^n$$

is a uniform limit on K of rational functions with pole at ∞, hence belongs to $B(E)$. Thus $B(E) \supset B(E_0)$ and we're done. □

PROOF OF RUNGE'S THM. We want to show that

if f is analytic in a neighborhood (say, U) of K, then $f|_K \in B(E)$.

Let γ be a closed path in $U \setminus K$ with winding number 1 about every point of K. (That this is feasible follows from the construction of a homology basis for the connected open set U earlier in this course.) Then $f|_K = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z} dw$ and Lemma V.A.7 gives the existence of R with poles in $(\gamma \subset) \mathbb{C}\backslash K$ such that $\|f - R\|_K < \epsilon$. But the fact that $B(E)$ is an algebra and $\frac{1}{z-\alpha} \in B(E)$ $(\forall \alpha \in \mathbb{C} \setminus K)$ by Lemma V.A.8 $\implies$ $R \in B(E)$. By taking ϵ smaller and smaller we get a sequence $R_n \to f$ uniformly on K, so that $f \in B(E)$ too. □

§ Mittag-Leffler's theorem.

We are now ready to generalize the construction of meromorphic functions from the first section to arbitrary open subsets of $\mathbb{C}$.

V.A.9. THEOREM. *Let $G \subset \mathbb{C}$ be an open set, $\{a_k\} \subset G$ a sequence of distinct points with no point of accumulation **in G**, and*

$$S_k(z) = \sum_{j=1}^{m_k} \frac{A_{jk}}{(z-a_k)^j}$$

the sequence of desired principal parts. Then there exists an $f \in \mathscr{Mer}(G)$ with polar set $\{a_k\}$ and principal parts S_k at a_k.

PROOF. By taking $K_n := \{z \mid d(z, \mathbb{C}\setminus G) \geq \frac{1}{n}\} \cap \overline{D}_n$, we get a sequence of compact subsets $K_n \subset G$ with

(a) $G = \bigcup_{n=1}^{\infty} K_n$,
(b) $K_n \subset \mathring{K}_{n+1}$, and
(c) each component of $\hat{\mathbb{C}} \setminus K_n$ containing a component of $\hat{\mathbb{C}} \setminus G$.

In each K_n there are *finitely* many $\{a_k\}$, and we define

$$I_1 := \{k \in \mathbb{N} \mid a_k \in K_1\}, \quad I_n := \{k \in \mathbb{N} \mid a_k \in K_n \setminus K_{n-1}\},$$

$$f_n(z) := \sum_{k \in I_n} S_k(z) \quad (n \geq 1).$$

Clearly f_n is rational and holomorphic in a neighborhood of K_{n-1}. Because of (c), Runge's Theorem yields the existence of a rational function $R_n(z)$ with poles in $\hat{\mathbb{C}} \setminus G$ and $\|f_n - R_n\|_{K_{n-1}} < (\frac{1}{2})^n$. Hence

$$f_1(z) + \sum_{k \geq 2} (f_k(z) - R_k(z))$$

has tails converging uniformly on each K_n (by the Weierstrass M-test with $M_k = (\frac{1}{2})^k$), and therefore on every compact subset of G. It thus defines a meromorphic function with the desired principal parts. □

§ Examples.

The Runge and Mittag-Leffler results are important, but they won't figure in here: we will simply revert to the first section (meromorphic functions on $\mathbb{C}$ with given principal parts) and apply the method of subtracting off a polynomial from each principal part.

V.A.10. EXAMPLE. In a neighborhood of zero we can write

$$f(z) = \frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{(\pi z - \frac{(\pi z)^3}{3!} + \dots)^2} = \frac{1}{z^2} + \frac{\alpha}{z} + \text{holo.}$$

But f even $\implies \alpha = 0 \implies \mathrm{PP}_0(f) = \frac{1}{z^2}$. By periodicity, the principal parts at $z = n \in \mathbb{Z}$ are $\mathrm{PP}_n(f) = \frac{1}{(z-n)^2}$. Moreover, $\sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}$ is convergent by comparison to $\sum \frac{1}{n^2}$ (essentially), and so

$$g(z) := f(z) - \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2} \in \mathscr{Hol}(\mathbb{C})!$$

Now g is periodic with period 1, and since

$$\begin{aligned} |\sin \pi z|^2 &= \frac{e^{\mathbf{i}\pi z} - e^{-\mathbf{i}\pi z}}{2\mathbf{i}} \frac{e^{-\mathbf{i}\pi \bar{z}} - e^{\mathbf{i}\pi \bar{z}}}{-2\mathbf{i}} \\ &= \frac{e^{-2\pi y} + 2 + e^{2\pi y} - e^{2\mathbf{i}\pi x} - 2 - e^{-2\mathbf{i}\pi x}}{4} \\ &= \cosh^2 \pi y - \cos^2 \pi x \ \longrightarrow\ \infty \quad \text{as } |y| \to \infty, \end{aligned}$$

g is bounded on vertical strips (say) of width 1. But then by periodicity, g is bounded on $\mathbb{C}$ hence constant by Liouville; and since the limit as $|y| \to \infty$ of both functions is zero, g must then be zero. Therefore

$$\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-n)^2}. \tag{V.A.11}$$

So no polynomial correct was required! In the next one we won't be so lucky:

V.A.12. EXAMPLE. The function

$$f(z) = \pi \cot(\pi z)$$

is already somewhat familiar. We want to reconstruct it from its principal parts $\mathrm{PP}_n = \frac{1}{z-n}$. What is for certain is that $\sum_{n\in\mathbb{Z}} \frac{1}{z-n}$ does *not* converge, because the harmonic series doesn't. Of course, we could be lazy and combine $\pm n$ terms, but let's not (at least, at first). Rather, we shall add the polynomial correction term $\frac{1}{n}$ to form

$$\sum_{n\in\mathbb{Z}}{}' \left(\frac{1}{z-n} + \frac{1}{n}\right) = \sum_{n\in\mathbb{Z}}{}' \frac{z}{n(z-n)},$$

where the prime means to omit $n = 0$. This is uniformly convergent on compact sets hence may be differentiated termwise to yield

$$-\sum_{n\in\mathbb{Z}}{}' \frac{1}{(z-n)^2} = \frac{-\pi^2}{\sin^2 \pi z} + \frac{1}{z^2}.$$

So we have

$$\begin{aligned}
\frac{d}{dz}[\pi \cot \pi z] = -\pi^2 \csc^2 \pi z &= \frac{-\pi^2}{\sin^2 \pi z} \\
&= \frac{d}{dz}\left\{\frac{1}{z} + \sum_{n\in\mathbb{Z}}{}' \left(\frac{1}{z-n} + \frac{1}{n}\right)\right\} \\
&= \frac{d}{dz}\left\{\frac{1}{z} + \sum_{n\geq 1} \left(\frac{1}{z-n} + \frac{1}{z+n}\right)\right\} \\
&= \frac{d}{dz}\left[\frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2-n^2}\right],
\end{aligned}$$

where both square-bracketed functions are odd. Hence they have zero constant term and

$$\pi \cot(\pi z) \underset{\text{on } \mathbb{C}}{=} \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \underset{\text{on } D_1}{=} \frac{1}{z} + \sum_{m\geq 1} a_{2m-1} z^{2m-1}$$

where

$$\begin{aligned}
a_{2m-1} = \frac{1}{2\pi \mathbf{i}} \oint_{|z|=\frac{1}{2}} \frac{\pi \cot(\pi z)}{z^{2m}} dz &= \sum_{n=1}^{\infty} \frac{1}{2\pi \mathbf{i}} \oint_{|z|=\frac{1}{2}} \frac{2z^{1-2m}}{z^2-n^2} dz \\
&= \sum_{n=1}^{\infty} \mathrm{Res}_0 \left(\frac{2/(z^2-n^2)}{z^{2m-1}}\right).
\end{aligned}$$

Now since $2/(z^2-n^2) = \frac{-2/n^2}{1-z^2/n^2} = -\frac{2}{n^2}\sum_{k\geq 0}\frac{z^{2k}}{n^{2k}}$, and Res picks out the term $k = m-1$, we obtain

$$a_{2m-1} = \sum_{n=1}^{\infty} \frac{-2}{n^2}\cdot\frac{1}{n^{2(m-1)}} = -2\sum_{n=1}^{\infty}\frac{1}{n^{2m}} = -2\zeta(2m).$$

But we computed in (III.D.12) that

$$a_{2m-1} = \frac{\pi^{2m}B_{2m}(-4)^m}{(2m)!}.$$

So we conclude that

$$\zeta(2m) = \frac{2^{2m-1}|B_{2m}|}{(2m)!}\pi^{2m}, \tag{V.A.13}$$

a rational multiple of π^{2m}.