Problem Set 1

Do problems 1 and 2. Then do 3 or 4, and 5 or 6. Use only the material we’ve covered so far in the notes. (You may use results stated but not proved, provided they aren’t exercises!)

(1) Let $M$ be a complex manifold. Assume $M$ is compact and connected, and let $f \in \mathcal{O}(M)$. Show that $f$ is constant.

(2) Let $V$ and $W$ be vector spaces (over the same field). Prove that $\bigwedge^n(V \oplus W) \cong \bigoplus_{p+q=n} \bigwedge^p V \otimes \bigwedge^q W$.

(3) In the following, $I$ is a multi-index, $x_j$ are local coordinates on a manifold, $\omega$ and $\mu$ are $p$- resp. $q$-forms, and $\xi$ is a vector field. Show the following (locally is ok):

(a) $\frac{\partial}{\partial x_j} dx_I = \begin{cases} 0, & j \notin I \\ (-1)^{\ell-1} dx_{I \setminus \{j\}}, & j = i_\ell \in I \end{cases}$

(b) $\xi_\omega(\mu \wedge \omega) = (\xi_\mu) \wedge \omega + (-1)^q \mu \wedge (\xi_\omega)$

(4) Check the formula (cf. page 6 of the notes)

$$d\omega(\xi^0, \ldots, \xi^p) = \sum_{0 \leq j \leq p} (-1)^j \xi_j \{\omega(\xi^0, \ldots, \hat{\xi}^j, \ldots, \xi^p)\} + \sum_{0 \leq j < k \leq p} (-1)^{j+k} \omega(\xi^j, \xi^k, \xi^0, \ldots, \hat{\xi}^j, \ldots, \hat{\xi}^k, \ldots, \xi^p)$$

in the special case where $\dim M = 2$ and $\omega$ is a 1-form (so $p = 1$). (Just do it in local coordinates $x, y$.)

(5) Make explicit and verify the formula $\Pi_* d\omega + d\Pi_* \omega = I_1^* \omega - I_0^* \omega$ from the notes (p. 8).

(6) Show that the (real) de Rham cohomology of $\mathbb{R}^2/\mathbb{Z}^2$ has (in degrees $0, 1, 2$) respective dimensions $1, 2, 1$. [Hint: the torus acts transitively on itself by translation. Use the Observation on p. 8 of the notes.]