Problem set 6

Page numbers refer to my notes. There are more problems in the notes; I’ve picked the most important ones.

(1) Let \((V, F)\) be a Hodge structure of weight \(n\), and \(U \subseteq V\) a subgroup. Prove that, together with this, the filtration \(U_C\) yields a (sub-)HS \(\iff U_C = \oplus_{p+q=n} V^{p,q} \cap U_C\). Show that this holds, in particular, for the kernel of a morphism of HS.

(2) Let \(V\) be an abelian group of even rank (OK to assume torsion-free), \(Q\) an alternating nondegenerate bilinear form on \(V\), \(n \in \mathbb{N}\) odd, \(\mathcal{h} = (h^{n,0}, h^{n-1,1}, \ldots, h^{0,n})\), and \(D\) the classifying space for HS on \(V\) of weight \(n\) and type \(\mathcal{h}\) polarized by \(Q\). Show that the real symplectic group \(G_R := \text{Aut}(V_R, Q)\) acts transitively on \(D\). Don’t try to do it in general at first: start with \(n = 1\). [Hint: every HS in \(D\) has a basis \(\{\omega_i\}\) subordinate to the Hodge decomposition and closed under complex conjugation, which can be taken to satisfy \(Q(\omega_i, \bar{\omega}_j) = \xi_i \delta_{ij}\) (where \(\xi_i = \pm \sqrt{-1}\)). Consider the transformation moving one such basis to another (for a different PHS in \(D\)); is it in \(G_R\)?]

(3) [to do after (2)] Prove in the setting of problem (2) that the isotropy subgroup \(H \subset G_R\) fixing a reference Hodge structure is in fact the product of unitary groups claimed on p. 182.

(4) Show that the inclusion of complexes \(\Omega_M^{\leq p} \hookrightarrow F^pA^*_M\) is a quasi-isomorphism. [Hint: use \(\bar{\partial}\)-Poincaré and (the total complex of) a double complex of sheaves.]

(5) Use the long-exact sequences associated to the two short-exact sequences on p. 189 to define a map \(d_1 : H^{p+q}(Gr_F^p C^\bullet) \to H^{p+q+1}(Gr^{p+1}_F C^\bullet)\), and check that \(d_1 \circ d_1 = 0\).

(6) Show that the infinitesimal period relation is trivial on the period domain for PHS of weight \(n = 2\) and Hodge numbers \(h^{2,0} = h^{0,2} = 1\) \((h^{1,1}\) arbitrary). (You don’t have to worry about the precise integral form of \(Q\), just that \(HR I/II\) hold.) [Hint: compute the tangent space to \(D\) and show, using the infinitesimal form of HR I for \(Q\) (cf. p. 205), that it is already contained in \(\text{Hom}(F^2, F^1/F^2) \oplus \text{Hom}(F^1, V/F^1)\).]

(7) Derive Example III.C.1 (p. 201) from the Picard-Fuchs equation in Problem Set 5 Exercise 8. (The function \(g(s)\) should be related to \(Q/\mathcal{P}\) in that problem.) Also, show how the “renormalized” version with parameter \(q\) arises from the family of complex 1-tori \(\mathbb{C}/\mathbb{Z}\langle 1, \tau\rangle\) over the upper half plane.