Definition 1: (i) A differentiable manifold \( M \) (of real dim. \( m \) and class \( C^k \)) is a (second countable Hausdorff) topological space with

\[
\left\{ \begin{array}{l}
\text{open cover } \{U_a\} \\
\text{homeomorphisms } \phi_a : U_a \to V_a \subset \mathbb{R}^m \quad \text{("charts")}
\end{array} \right.
\]

s.t. \( \phi_a \circ \phi_b^{-1} : \phi_b(U_b) \cap \phi_a(U_a) \to \phi_a(U_a) \) ("transition functions")

is a \( C^k \) diffeomorphism from one open \( \phi B \) to open \( \phi A \).

(ii) For \( U \subset M \) open and \( 0 \leq s \leq k \),

\[ C^s(U, \mathbb{R}) = \{ \text{functions } f : U \to \mathbb{R} \mid f \circ \phi_a^{-1} \in C^s(\phi_a(U_a \cap U)) \} \]

(iii) Morphisms \( C^s(M, N) \) between manifolds defined in same vein.

(iv) \( M \) is \{ orientable \( \iff \) atlas with all \( \det (D\phi_{ab}(p)) > 0 \) \}

A derivative of \( p \in M \) is any linear functional \( D : C^k(M, \mathbb{R}) \to \mathbb{R} \)

satisfying the "Leibniz rule" \( D(fg) = f(p)D(g) + g(p)D(f) \).

Let \( D_C(M) \) denote the set of these—it is a vector space \( / \mathbb{R} \). Moreover, if \( (U, \varphi) \) is a small nbhd. and \( f|_U = f|_V \), taking \( g|_M = 1 \) and (on a smaller nbhd.) \( g|_V = 0 \),

\[
D(f|_U - f|_V) = D(g)(f|_U - f|_V) = (D(g)(f|_U)) + (D(g)(f|_V)) = 0
\]

also \( D(1) = D(1, 1) = 2(1 \cdot D(1)) = D(1) = 0 \).

Writing \( \varphi(x) = (x_1, \ldots, x_m) \), note that \( D_0 \varphi^* \in \text{Der}_o(\mathbb{R}^m) \). Any

\[
g \in C^k(M, \mathbb{R}) \quad \text{such that } \quad \varphi_0 \in \text{Der}_o(\mathbb{R}^m)
\]

\[
g(x) - g(x) = \int_0^1 \frac{d}{dt} g(\varphi(t)) \, dt = \int_0^1 E \frac{\partial}{\partial x_j} (t \varphi) \, dt
\]

* for \( U \) not open, it's the same thing, with \( C^k \) extension to some open \( \supset U \).
\[ g(x) = g(0) + \sum x_i \theta_i(x), \quad \theta_i(0) = \frac{\partial g}{\partial x_i}(0) \]

\[ \bar{D}(g) = \sum_1^m \partial_i (g) \cdot \theta_i(x) + \sum_1^m \partial_i (g) \cdot x_i \phi_i(0) - \sum_1^m \partial_i (x^i) \cdot \frac{\partial g}{\partial x_i}(0) \]

\[ \bar{D} \text{ is a directional derivative at } 0. \text{ So} \]

\[ \text{Def}_p(M) \cong \mathbb{R}^{\frac{n}{\partial x_1, \ldots, \partial x_m}} \text{ is an } m \text{-dim. vector space} \]

and we set

\[ T_pM := \text{Def}_p(M), \quad T^{(v)}_M := \bigcup_{p \in M} T^{(v)}_p \]

(\text{co-tenent bundles})

To "topologize" these, we turn to the

\textbf{Definition 2:} (i) A \textit{real vector bundle} \((\text{of rank } r \text{ and class } C^\infty)\) over \(M\) is a (separable, Hausdorff) topological space \(E\) together with a map \(\pi : E \to M\), such that

- the fibers \(E_x = \pi^{-1}(x)\) are real vector spaces of dim. \(r\)
- \(E\) is over \(\{U_{\alpha}\}\) and class \(C^\infty\) homom. \(\Phi_{\alpha} : \pi^{-1}(U_{\alpha}) \to \mathbb{R}^r \times M\)

\[ \text{(a)} \quad p_{\alpha \beta} \circ \Phi_{\alpha} = \pi \big|_{U_{\alpha \beta}} \]

\[ \text{(b)} \quad \forall x \in U_{\alpha}, \quad p_{\alpha \beta} \circ \Phi_{\alpha} \circ \pi^{-1}_{E_{\alpha \beta}} : E_{\alpha \beta} \to \mathbb{R}^r \text{ is an } E \text{ of } \mathbb{R} \text{-vector space.} \]

(ii) A (class \(C^\infty\)) \textit{morphism} \(\Psi : E \to F\) of \(\text{v.b.'s}\) \(E / M \to F / M\) is a (class \(C^\infty\)) morphism of the underlying manifolds s.t.

- \(\Psi\) is linear on fibers
- \(\pi \circ \Psi = \pi\)

A v.b. is \textit{trivial} if \(E \cong M \times \mathbb{R}^r\).

From the definition, it follows that the \(\Phi_{\alpha \beta} : U_{\alpha \beta} \to GL(r, \mathbb{R})\) are actually \(C^\infty\) functions.

\[ \Phi_{\alpha \beta} : U_{\alpha \beta} \to GL(r, \mathbb{R}) \]

\[ \text{s.t. } \Phi_{\beta \delta} \circ \Phi_{\alpha \beta} = \Phi_{\alpha \delta} \text{ on } U_{\alpha \beta \delta} \]

\[ \text{(conform condition)} \]

Moreover, \(E\) is a \(C^\infty\) manifold, and we can perform all the operations of linear algebra on the \(\Phi_{\alpha \beta}\) hence on \(E\) — this yields bundles \(\lambda^*E, E^\vee, \text{ etc.}\)

\textbf{Proposition 1:} \(T_M\) is a \(C^\infty\) bundle over \(M\).
Proof: The essential point here is that

\[ \Phi_B = (\Phi_B)_\ast \left( \begin{pmatrix} \frac{\partial (x^1, \ldots, x^n)}{\partial (x^1, \ldots, x^n)} \end{pmatrix} \right) \quad \text{in base}. \]

\[ \blacksquare \]

Definition 3: A \((C^1)\) section of a vector bundle, written \( s \in C^1(M, E) \), is a \((C^1)\) morphism \( \sigma : M \to E \) s.t. \( \pi \sigma = id_M \).

Example: sections of \( \left\{ \frac{T_m}{\Lambda^1 M} \right\} \) are called \( \left\{ \text{vector fields} \right\} \) different potentials.

Definition of Lie bracket: Consider \( \mathcal{F} = \sum_{j=1}^n \xi_j \partial_{x_j} \in T_{\pi(p)} \),

\[ f \in C^1(p, \mathbb{R}) \]

The \( df \mathcal{F} (p) := \mathcal{F}(f) = \sum_{j=1}^n \xi_j \partial_{x_j} (f) \) defines \( df \mathcal{F} \in T_{\pi(p)} \).

In particular, \( dx^i (\mathcal{F}) = \xi_i \Rightarrow \{ dx^i \} = \{ \partial_{x_j} \} \) is called \( \{ \partial_{x_j} \} \) and \( T_{\pi(p)} \),

\[(B.2) \quad \text{and} \quad df = \sum_{j=1}^n \xi_j \partial_{x_j} \in C^1(p, T_{\pi(p)}) \]

Lie bracket: consider \( \xi, \eta \in C^1(U, T_m) \)

\[ [\xi, \eta] (f) := \xi(\eta(f)) - \eta(\xi(f)) \]

\[ \in C^1(U, T_m) \]

In coordinates, \( [\xi, \eta] = \sum_{1 \leq j, k \leq n} \left( \xi_j \frac{\partial \eta_k}{\partial x_j} - \eta_j \frac{\partial \xi_k}{\partial x_j} \right) \partial_{x_k} \)

Wedge/contraction: given \( \omega \in C^1(M \wedge^n T_m) \) \( p \)-form

\[ \sum \left. \omega_{I_1} \wedge \cdots \wedge d\omega_{I_p} \right|_{I_1=\ldots=I_p} \quad \text{and} \quad \xi \in \text{vector field} \]

\[ \wedge \text{ is a } (p+q) \text{-form} \]

\[ \wedge \text{ is a } (p-1) \text{-form} \]

Example:

\[ \frac{\partial}{\partial x_j} \wedge dx_I = \begin{cases} 0, & \text{if } j \notin I \\ (-1)^{i-1} dx_{i-1} \wedge \cdots \wedge dx_j, & j = i \in I \end{cases} \]

\[ \wedge \mathcal{F}_\ast (\mathcal{F} \wedge \omega) = (\mathcal{F} \wedge \omega) - \wedge (\mathcal{F} \ast \omega) \]

\[ / / \]
Exterior derivative: \( d \omega := \sum_{0 \leq i < m} \frac{\partial \omega}{\partial x_i} dx_i \wedge \cdots \wedge dx_m \) defines
\[ d : C^\infty(M, \Lambda^p T^*_m) \to C^{\infty}(M, \Lambda^{p+1} T^*_m) \]

Ex: For \( \omega^0, \ldots, \omega^p \) vector fields,
\[
\omega^0(\omega_0, \ldots, \omega_p) = \sum_{0 \leq i \leq p} (-1)^i \omega_i(\omega_0, \ldots, \hat{\omega}_i, \ldots, \omega_p)
+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \omega_i(\omega_j, \omega_0, \ldots, \hat{\omega}_i, \ldots, \hat{\omega}_j, \ldots, \omega_p)
\]
Show \( d \omega \) is intrinsically defined.

We also have \( d(m \omega) = d\omega \wedge m \omega \)
and \( (e \geq 2) \) \( d^2 = 0 \) \( \Leftarrow \) equality of 2nd power:
\[
\begin{aligned}
    e \cdot f, & f(df) = d(e \cdot f, df, \ldots, df) \\
    d(e, df, df, \ldots, df) & = d(f, df, df, \ldots, df) \\
\end{aligned}
\]
Pullback: given \( F \in C^\infty(M, N) \), 
\[ \omega = \sum_{j} \omega_j(x) dy_j \in C^\infty(N, \Lambda^m T^*_N) \]
define the pullback \( F^* \omega := \sum_{j} F_j^*(x) df_j(x) \wedge \cdots \wedge df_{m-1}(x) \).
The chain rule is (for \( G \in C^\infty(M', M) \)) \( G^* F^* \omega = (F \circ G)^* \omega \),
and shows this is invariant w.r.t. choice of coordinates. We also have \( d\circ F^* = F^* \circ d \).

De Rham cohomology: set \( K^p := C^\infty(M, \Lambda^p T^*_m) \). Since \( d^2 = 0 \),
\[ K^p := \cdots \to K^{p+1} \xrightarrow{d} K^p \xrightarrow{d} K^{p-1} \to \cdots \]
is a complex \((\text{im } d^{p+1} = \ker d^p)\).
We define
\[ H^p_{dR}(M, \mathbb{R}) := \ker d^p / \text{im } d^{p-1} = \text{closed forms} / \text{exact forms} \]
By \((1)\) we have \( F^* : H^p_{dR}(N, \mathbb{R}) \to H^p_{dR}(M, \mathbb{R}) \).

\((H^p_{dR}(M, \mathbb{R}) \text{ is obtained by doing everything w/ compact support}), \)
for \( M \) noncompact this gives a different result.

So this's your differential calculus.
Now we briefly review a bit of integral calculus.

**Definition 4:** Let \( \omega \in \mathcal{C}_c^0(U, \Lambda^\bullet T^*_M), \ U \subset M. \)

Then \[
\int_M \omega = \int_{\mathcal{R}^m} \left( \frac{\partial f}{\partial x_1} \right) dx_1 \wedge \cdots \wedge dx_m
\]
is independent of coordinates by the \( \Delta \) or wedge formula.

If \( U \) is replaced by \( M \), and \( \{ g_\lambda \} \) is a partition of unity (\( \sum g_\lambda = 1 \)),

\[
\int_M \omega = \sum_{\lambda} \int_{\mathcal{R}^m} g_\lambda \omega
\]
is well-defined.

Fe \( \mathcal{C}(N\setminus M) \) will

\[
(\pm) \int_N \omega \quad \text{with sign acc. to whether } \text{det } (\text{Jac}_\omega) > 0.
\]
(i.e. orientation preserved)

** Stokes theorem:** Let \( \omega \in \mathcal{C}'(M, \Lambda^\bullet T^*_M) \), and

\( \Gamma \subset M \) be a compact subset with piecewise \( \mathcal{C}' \) boundary.

Then \[
\int_{\partial \Gamma} \omega = \int_{\Gamma} \mathcal{H} \omega.
\]

**Sketch of pr.:** locally \( \omega = \sum f_j \, dx_1 \wedge \cdots \wedge dx_m \)

\[
\int_{\partial \Gamma \cap U} \omega = \int_{\Gamma \cap U} \mathcal{H} \omega = 0 \quad \text{on } U
\]

If \( \omega \) compactly supported on \( U \),

\[
\int_{\partial \Gamma \cap U} \omega = \int_{\mathcal{R}^m} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_m
\]

(partially \( f \) wrt \( x_j \))

\[
\int_{\partial \Gamma \cap U} \omega = \sum (-1)^{j-1} \int_{\mathcal{R}^m} \frac{\partial f_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_m
\]

General result follows from a partition of unity argument.

Now let \( \Gamma : [0,1] \times M \to M, \ I_t : M \to [0,1] \times M \)

\( t \mapsto (t, p) \quad p \mapsto (t, p) \)

and define
\[ \Pi_k : C^k([0,1] \times M, \Lambda^r T^\vee M) \to C^s(M, \Lambda^p T_M^\vee) \]

by integrating along the fiber. (This is called “push-forward,”
and kills all terms of type dx_k – only dx_k’s go somewhere.)

\[ \Pi_k^* dw + d\Pi_k^* w = I_1^* w - I_0^* w \quad (\text{essentially FTC with cancellation}) \]

**Observation:** If \( F, G \in C^\infty(M, N) \) are smoothly homotopic
\( (\exists H \in C^\infty([0,1] \times M, N) \text{ s.t. } H_0 I_0 = F, H_1 I_1 = G) \)

then \( F^* w - G^* w = (I_0^* - I_1^*) (H^* w) \)
\[ = \Pi_k^* dH^* w + d(\Pi_k^* H^* w) \]
\[ \text{exact} \]
\[ \Rightarrow F^* = G^* : H^{p\mu}(N, \mathbb{R}) \to H^{p\mu}(M, \mathbb{R}) \]

**Example:** \( N = M \) contractible, \( F(M) = \{ x_0 \} \), \( G = I^\mu M \)
\[ \Rightarrow H^{p\mu}(M, \mathbb{R}) = \begin{cases} \mathbb{R}, & p > 0 \\ 0, & p < 0 \end{cases} \]

As a corollary, we obtain the

**d-Poincaré Lemma:** Given an \( \mathbb{R}^n \) stratified open set (\( \Rightarrow \) contractible),
\( \omega \in C^s(U, \Lambda^p T_U^\vee) \) closed \( \Rightarrow \omega \) exact. \( (\neq \partial \eta) \)

We conclude with some further examples of de Rham cohomology groups.
Example 3: \( \dim H^0_{dR}(M, \mathbb{R}) = \# \) connected components of \( M \).

Proof: \( H^0_{dR} = \ker(d^{(0)}) = \text{constant functions} \).}

Example 4: \( H^2_{dR}(S^2, \mathbb{R}) \cong \mathbb{R} \)

Proof: Let \( \tau : S^2 \to \mathbb{R}^3 \) (counting \( x_1, x_2, x_3 \))
\[ \Theta = \sum x_j \frac{\partial}{\partial x_j}. \]
\[ \rho = dx_1 + dx_2 - dx_3; \]
then \( \omega = \theta \cdot \rho \) is \{ \begin{cases} \text{nonzero} \\ \text{closed (since of top degree)} \\ \text{SO(3,\mathbb{R})-invariant} \end{cases} \}
and is the unique such form (up to scale).
If \( \omega = dw, \) for \( w \in C^\infty(S^2, T_{S^2}^*), \) then for \( g \in \text{SO}(3,\mathbb{R}) \)
\( g^\ast\omega = g^\ast(dw) = d(g^\ast w) \)
\[ \Rightarrow \omega = d\left( \int_{\text{SO}(3,\mathbb{R})} g^\ast \omega \, d\mu \right) \]
\[ \text{Hausdorff measure:} \int S^2 \, d\mu = 1, \quad g^\ast d\mu = d\mu \text{ (V)} \]
\[ \Rightarrow \omega = d(0) = 0, \quad \text{a contradiction}. \]

Example 5: using a group action one can also show
\( a) \) \( H^1_{dR}(S^2, \mathbb{R}) = \mathbb{R} \)
\( b) \) \( \dim H^m_{dR}(\mathbb{R}/\mathbb{Z}, \mathbb{R}) = \begin{cases} 1, & m = 0 \\ 2, & m = 1 \\ 1, & m = 2 \end{cases} \)

Example 6: If \( M \) is compact & connected, then \( \dim H^m_{dR}(M, \mathbb{R}) = 0 \) or 1,
and \( M \) orientable \( \iff \dim H^m_{dR}(M, \mathbb{R}) = 1 \).

Ex: use a partition of unity \& Stokes theorem to prove:
\( M \) orientable \( \Rightarrow H^m_{dR}(M, \mathbb{R}) \neq 0 \).