C. Complex manifolds

We begin with a modest dose of "several complex variables".

Let \( \Omega \subseteq \mathbb{C}^n \) be a connected open set, consider \( z_j = x_j + iy_j \), \( f : \Omega \to \mathbb{C}, \quad f = u + iv \). (as assumptions on the function \( f \))

**Definition 1:** (a) \( f \) analytic \( \iff \forall \ z \in \Omega \ F \text{ open } U \subseteq \Omega \)

\[ f(z) = \sum_{k=0}^{\infty} c_k(z - z_0)^k \]

(Convergent in \( U \))

(b) \( f \) holomorphic \( \iff f \) separately holomorphic in each \( z_j \)

(with other \( z_j \) held fixed)

**Hartogs theorem:** (a) \( \iff \) (b)

\((b \to a)\) easy

**Sketch:** Assume \( f \) locally bounded (otherwise it's quite hard), iteratively applying 1-variable Cauchy gives

\[ f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{f(\xi)}{\prod_{j=1}^{n} (\xi_j - z_j)} \prod_{j=1}^{n} d\xi_j \]

(The local boundedness comes in at the end, in rearranging the \( \frac{1}{\cdot} \)s.)

Expanding the \( \frac{1}{\xi_j - z_j} = \frac{1}{\xi_j} \) in power series completes the proof.

More generally, if \( f \) is just \( C^1 \), then we write

\[ df = \sum \frac{\partial f}{\partial x_j} \, dx_j + \sum \frac{\partial f}{\partial y_j} \, dy_j = \sum \left[ \frac{\partial f}{\partial x_j} \, dx_j + \frac{\partial f}{\partial y_j} \, dy_j \right] + \sum \left[ \frac{\partial f}{\partial x_j} \, dx_j + \frac{\partial f}{\partial y_j} \, dy_j \right] \]

which suggests "\( \bar{\Omega} = \Omega + \bar{\Omega} \)" (more on this in a moment).
We may view $df_p$ as the map $f^*_p : T_p \rightarrow T_{f(p)} \mathbb{R}^2$ given w.r.t. bases $\left\{ \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j} \right\}$ and $\left\{ \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\}$ by

$$
\begin{pmatrix}
u_x & u_x & u_y & v_x & v_y \\
u_y & u_x & u_y & v_x & v_y
\end{pmatrix}
$$

This element of $\text{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ is $C$-linear, i.e., comes from $\text{Hom}_C(C^n, C)$.

If the metric is of the form

$$
(\alpha, \beta, \ldots, \alpha, -\beta)
$$

(i.e., CR equations hold).

We conclude that

$$
f \in C^0(\Theta) \iff \int f = 0 \iff f^*_p \text{-linear} \quad (v_p \in \Theta)
$$

Besides (C.1-2) there are a lot of

\textbf{similarities w.r.t. 1-variable case:}

(i) $\{f_k\} \subset C^0(\Theta)$ locally uniformly bounded $\Rightarrow$

\text{normally convergent subsequence $\{f_k\}_k$ and then lim $f_k \in C^0(\Theta)$}.

(ii) $U \subset \Theta$ open nonempty

$$
f \in C^0(U), \quad f^*_{|U} \subset C
$$

f constant \quad [\text{see Courant - see Vol. III}]

(iii) $f \in C^0(\Theta)$, $z \in \Theta$

$$
|f(z)| \geq |f(\bar{z})| \quad \forall \bar{z} \in \Theta
$$

(iv) $f \in C^0(\Theta \setminus \{z \in \mathbb{C} \})$

f bounded in nbhd. of $\{z \in \mathbb{C} \}$

\textbf{Proof for (iv):} \text{near } (0, 0) \in \Theta \text{ (say)},

$$
f(z, \bar{z}) = \sum_{j=0}^\infty a_j z^j \bar{z}^j
$$

where $a_j \neq 0$ is holomorphic by monotone (1-var.).

We then get

$$
|z| \leq \sum_{j=0}^\infty |a_j| |z|^j \leq \sum_{j=0}^\infty |a_j| |w|^j = 0
$$

\text{for boundedness now, which } a_j u \text{ for } j < 0.

\textbf{\textit{\textcircled{C}}}

What's different:

(v) no analogue of Riemann mapping thm. — (unit disk) \( \cong \mathbb{E} \) (unit n-ball)
are not biholom. [Poincaré]

(vi) largest domain of convergence of a power series not a polygon
(or ball) in general

(ni) Hartogs' phenomenon: \( n \geq 2 \) only

\[ K \subset \mathbb{B} \text{ compact} \]
\[ \mathbb{B} \setminus K \text{ connected} \]
\[ f \in O(\mathbb{B} \setminus K) \]

Pf. for \( n=2 \), \( K=\{(0,0)\} \)

\[ \frac{d^j}{dz^j} f(z) = 0 \text{ for } j < 0 \] \( z \to 0 \), since \( f \) is a holomorphic function of \( z \) on a disk about 0. By (ii), \( d^j < 0 \Rightarrow 0 \).

1-variable \( z \) - Poincaré Lemma: Let \( K \subset \mathbb{C} \) be compact w/precursors

\( C \)-bounded, \( f \in C'((K, \mathbb{C})) \)

Then by Stokes' theorem, for \( \tau \in \text{int}(K) \)

\[ \int_{\partial K} \frac{\partial f/\partial \bar{z}}{w-z} \ d\bar{z} = \frac{1}{2\pi i} \int_{\partial \mathbb{C}} \frac{f(w)}{w-z} \ dw + \frac{1}{2\pi i} \int_{\partial K} \frac{f(w)}{w-z} \ dw 
\]

which gives a generalization of Cauchy. If \( f \) is compactly supported

in \( K \), then the \( \int_{\partial K} f = 0 \), and this becomes

\[ f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{C}} \frac{\partial f/\partial \bar{z}}{w-z} \ dw \ d\bar{z} = \frac{1}{2\pi i} \int_{\partial \mathbb{C}} \frac{f(w)}{w-z} \ dw \ d\bar{z} + \frac{1}{2\pi i} \int_{\partial K} \frac{f(w)}{w-z} \ dw \ d\bar{z} \]

We conclude the

Theorem 1: Given a 1-form \( f(z) \, dz \) (of class \( C^1 \)) compactly supported in \( K \),

\[ F \in C^1(K, \mathbb{C}) \text{ satisfying } \overline{\partial} F = f(z) \, dz \]
Finally, we have the

**Holomorphic Inverse Function Theorem**: Let \( U, V \subset \mathbb{C}^n \) and \( F \in \text{Hol}(U, V) \), i.e., \( F_j \in \mathcal{O}(U) \). If \( F_j \bigg|_{p} \) is nonsingular (det \( F_j \bigg|_{p} \neq 0 \))

then \( \exists \ Y \subset U, V \subset V \) s.t. \( F|_Y : U \to V \) is biholomorphic. \( \Box \)

**Remark**: More generally, if \( V \subset \mathbb{C}^m \) and \( F_j \bigg|_{p} \) has rank \( k \) everywhere, the rank theorem says that (up to a local biholomorphic change of coordinates) \( F \) takes the form \( (z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0) \) locally.

Now we can get to business: the holo. analogue of Defs. B.1-2.

**Definition 2**: (a) A complex manifold \( M \) of (complex) dimension \( n \) is a differentiable manifold (of real dim. \( 2n \)) having biholomorphic

transition functions \( \Phi_{\alpha \beta} \). In particular, all \( (\Phi_{\alpha \beta})_p \) are \( \mathbb{C} \)-linear.

(b) \( \Omega(M) \) and \( \text{Hol}(M, N) \) are also easy tweaks to the

previous definitions.

(c) A holomorphic vector bundle \( E \twoheadrightarrow M \) of (complex)

rank \( r \), is defined as in Def. B.2, but

\[
\left\{ \begin{array}{l}
\text{taking } \Phi_{\alpha} : \pi^{-1}(U_\alpha) \iso U_\alpha \times \mathbb{C}^r \text{ to be biholomorphic} \\
\text{taking } \pi_\mathbb{C} \circ \Phi_{\alpha} : E_p \twoheadrightarrow \mathbb{C}^r \text{ to be } \mathbb{C} \text{-vector spaces}.
\end{array} \right.
\]

Equivalently, the transition functions \( \Phi_{\alpha \beta} \) map \( U_{\alpha \beta} \twoheadrightarrow \text{GL}(r, \mathbb{C}) \)

and are holomorphic. \( \Box \)

# obviously, we have to identify the \( \mathbb{R}^{2m} \supset V_\alpha \) with \( \mathbb{C}^n \) for this to make sense
Remark 2: Any such $M$ is orientable, since
\[ \det (\phi_p) = \det \left( \begin{array}{cc} A & -B \\ B & A \end{array} \right) = |\det (A + B)|^2 > 0. \] (cf. (A.5)).

Explain that for $M$ compact, convex, connected and $f \in C(M)$, $f$ is constant.

Given a complex manifold $M$, the "underlying" real manifold has a tangent bundle (of rank $2n$). This has the structure of a holomorphic vector bundle $T_m$, since $\phi_{p\tilde{p}} = (\phi_p)_{\tilde{p}}$ are complex-linear (linear, on complex-linear (planar) and have holomorphic matrix coefficients. By (A.4) we can think of this equivalently as the underlying real bundle $T^R_m$ together with $J \in C^\infty(M, \text{End}(T^R_m))$ satisfying $J^2 = -\text{id}$ (i.e. $(T^R_m) \otimes T^R_m$).

The chart $\phi_{\tilde{p}}$ gives (local) coordinates $(z_1, \ldots, z_n)$ on $U \subset M$. As before we have the identification:
\[ (T_m)_{\tilde{p}} \cong \mathbb{R} \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \rangle. \]

Though this has a complex structure, we don't have $\frac{\partial}{\partial z_1} \frac{\partial}{\partial \bar{z}_1}$. To get this we must consider the "complexified tangent bundle" $T_m \otimes \mathbb{C}$ of complex rank $2n$.

\[ T^R_m \otimes \mathbb{C} = \mathbb{C} \langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n} \rangle. \]

(C.3)
\[ = \mathbb{C} \langle \frac{\partial}{\partial z_1}, \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n}, \frac{\partial}{\partial \bar{z}_n} \rangle. \]

\[ = T^R_{m, \tilde{p}} \oplus T^R_{m, \tilde{p} \tilde{p}}. \]
Proposition 1: \( T^0 \) is a \( (\text{rank } n) \) holomorphic vector bundle.

Proof: \( v \rightarrow v - iJ(v) \) induces an isomorphism \( T^0 \cong T^0 \) (It is \( \mathbb{C} \)-linear since \( J(u) \rightarrow J(u) - iJ(J(u)) = J(u) + iv \) \( \mathbb{R} \)-linear by \( i \) on \( T^0 \)).

By duality we have a decomposition

\[
T^0 \otimes \mathbb{C} \cong (T^0 \otimes \mathbb{C})^* \cong \left( T^0 \otimes \mathbb{C} \right)^*(\mathbb{C}, \mathbb{C}) \cong T^0(1, 0) \oplus T^0(0, 1)
\]

In coordinates:

\[
\begin{align*}
\frac{\partial}{\partial z_k} \frac{\partial}{\partial \bar{z}_k} & = \delta_{jk} \quad \frac{\partial}{\partial z_k} \frac{\partial}{\partial \bar{z}_k} = 0 \\
\frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial z_k} & = 0 \quad \frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_k} = \delta_{jk}
\end{align*}
\]

Another reason why you'd want to introduce \( T^0 \otimes \mathbb{C} \): we have

\[
d : C^\infty(M, \mathbb{C}) \rightarrow C^\infty(M, T^0 \otimes \mathbb{C}) \]

To differentiate \( C^\infty(M, \mathbb{C}) \) functions, we have no choice! That is,

\[
(C.5) \quad \boxed{d : C^\infty(M, \mathbb{C}) \rightarrow C^\infty(M, T^0 \otimes \mathbb{C})}
\]

In coordinates this allows us to write (computing \( d \) on \( p \). p. 10)

\[
(C.6) \quad d f = \sum \frac{\partial f}{\partial z_j} \frac{\partial}{\partial z_j} + \sum \frac{\partial f}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_j}
\]

Now strictly:

Definition 3: The elements of

\[
A^k(M) := C^\infty(M, \Lambda^k(T^0 \otimes \mathbb{C}))
\]

are called \textit{complex-valued differential} \( k \)-forms on \( M \).

Now by (A.1),
\[ \Lambda^k(T^*_M \otimes \mathbb{C}) = \bigoplus_{p^q + k} \Lambda^p \bigwedge^q (T^*_M \oplus \Lambda^p \bigwedge^q T^*_M) \]

and accordingly

\[ A^k(M) = \bigoplus_{p^q + k} \bigwedge^p \mathcal{C}^\infty(M, \Lambda^p \bigwedge^q T^*_M) = \bigoplus_{p^q + k} A^{p,q}(M) =: \text{Forms of type } (p,q) \]

A given \( \omega \in A^{p,q}(M) \) may be written locally (in \( U \subset M \)) as

\[ \sum_{I,J} \omega_{I,J} (x) \, dx_I \wedge dx_J \quad , \quad \omega_{I,J} \in \mathcal{C}^\infty(U, \mathbb{C}) \]

Now (6.6) shows that on functions, the exterior derivative splits into \((1,0)\) and \((0,1)\) parts; viz., for \( f \in \mathcal{C}^\infty(M, \mathbb{C}) \),

\[ df = df + \overline{\delta f} \]

and

\[ f \in \mathcal{C}^\infty(M) \iff \overline{\delta f} = 0 \iff df \in \mathcal{C}^\infty(M, \bigwedge^1 T^*_M) \]

in which case \( df \) is actually a holomorphic section of \( \bigwedge^1 T^*_M \). This splitting \( d = \partial + \overline{\delta} \)

can be extended to \( \mathbb{C} \)-valued forms:

\[ \partial \omega := \sum_{I,J} \sum_{I < J} \frac{\partial \omega_{I,J}}{\partial \overline{z}_k} \, dz_k \wedge dz_I \wedge dz_J \]

\[ \overline{\delta} \omega := \sum_{I,J} \sum_{I > J} \frac{\overline{\delta \omega_{I,J}}}{\partial \overline{z}_k} \, dz_k \wedge dz_I \wedge dz_J \]
Giving maps

\[ \begin{align*}
\delta : & \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p+1,q}(M) \\
\overline{\delta} : & \mathcal{A}^{p,q}(M) \to \mathcal{A}^{p,q+1}(M),
\end{align*} \]

and the

\[ 0 = \delta^2 = (\delta + \overline{\delta})^2 = \delta^2 + (\delta \overline{\delta} + \delta \overline{\delta}) + \overline{\delta}^2, \]

so that (in our first argument "by type")

\[ \delta^2 = (\delta \overline{\delta} + \delta \overline{\delta}) = \overline{\delta}^2 = 0. \]

Remark 3: It's useful to have the following picture of what it means for a \( k \)-form \( \omega = \sum_{\mu=1}^{\nu} \omega_{\mu} \) to be closed:

We must have \( \delta \omega_{\mu} = 0 = \overline{\delta} \omega_{\mu} \) and \( \delta \omega_{\mu} = -\overline{\delta} \omega_{\mu+1} \) for each \( (p,q) \).

We also have the following analog of de Rham cohomology:

\[ \begin{align*}
K^{p,q}_{-1} & := \mathcal{A}^{p,q}_{-1}(M) \\
\downarrow \delta_{-1} & \\
K^{p,q} & := \mathcal{A}^{p,q}(M) \\
\downarrow \delta_q & \\
K^{p,q}_{+1} & := \mathcal{A}^{p,q}_{+1}(M)
\end{align*} \]

We have
**Definition 4**: Dolbeault cohomology is
\[ H^{p,q}_{\overline{\partial}}(M, \mathbb{C}) = \frac{\ker \overline{\partial}^q}{\text{im } \overline{\partial}^{q-1}} \]

\[ L^p(M) := \Theta(M, \Lambda^p T^* M) \]

**Lemma** (C.13)

For holomorphic forms, there are locally of the form \( \sum w_I(z) dz_I \),
\( w_I \in \Theta(U) \) — we prove the

**Proposition 2**: \( L^p(M) \subseteq H^{p,0}_{\overline{\partial}}(M, \mathbb{C}) \) (\( \forall p \))

**Proof**: Given \( \omega \in L^p(M) \), \( \overline{\partial} \omega = 0 \).

Conversely, let
\[ \omega = \sum_{I} w_I(z) dz_I, \quad w_I \in \Theta(U) \]
\[ \frac{\partial}{\partial \bar{z}_j} \omega = 0, \quad j = 1, \ldots, n \]

\[ \Rightarrow \frac{\partial w_I}{\partial \bar{z}_j} = 0 \]

\[ \Rightarrow w_I \in \Theta(U) \]

So \( \omega \in L^p(M) \). Finally, note that \( \omega = \overline{\partial} \beta \) is possible

"by type".

**Remark 4**: It is also true that \( L^p(M) \subseteq H^{p,0}_{\overline{\partial}}(M, \mathbb{C}) \). But

without the Hodge theorem all we can show is that \( \omega \in L^p(M) \)
defines a class in \( H^{\cdot,\cdot}_{\overline{\partial}}(M, \mathbb{C}) \) (not even that the resulting
map \( L^p(M) \to H^{p,0}_{\overline{\partial}}(M, \mathbb{C}) \) is injective). This is done by observing
(1) \( \dd w = 0 \) and (since wedging \((n+1)\) \(dz\)'s together gives 0) \( \dd w = 0 \) 
\[ \Rightarrow \dd w = 0. \]

\[ \text{Remark 5: Given } F \in \text{Map}(M, N) \text{, } F^* : H^{p,q}_{\text{dR}}(N) \rightarrow H^{p',q'}_{\text{dR}}(M) \]

is defined. This is because \( \int F^* dw = dF^* \) is in \( T^\ast \Phi, \) 
\[ \int F^* \dd \omega = dF_k \quad \text{is in } T^\ast (\Phi) \]

\[ \Rightarrow F^* \text{ preserves type } (\text{FALSE for } C^\infty F). \]
So, for a form \( \alpha \)

\( (C^\infty \text{ of type } (p,q)) \)

\[ \delta F^* \alpha + \bar{J} F^* \bar{J} \alpha = dF^* \alpha = F^* \dd \alpha = F^* \delta \alpha + F^* \bar{J} \alpha \]

\( (p+1, q) \quad (p,q+1) \)

\[ \Rightarrow F^* \text{ commutes with } \delta, \bar{J}. \]

Ex. Show that the cup products on Deligne cohomology (defined by wedging forms together) are well defined.

We now come to the key result of this section:

**Theorem 2 (\( \bar{J} \)-Poincaré Lemma):** Let \( U \subset C^\infty \) be a polydisk, \( q \geq 1, \) and \( \alpha \in A^{p,q}(U) \) with \( \dd \alpha = 0. \) Then \( \exists \)

\[ \beta \in A^{p,q+1}(U) \text{ s.t. } \dd \beta = \alpha. \] (In other words, \( H^{p,q+1}_{\text{dR}}(U) = 0 \) for \( q \geq 1. \)

**Proof:** It will be convenient to choose a nested covering of \( U : \)

\[ U = \cup_{k=1}^{\infty} U_k, \quad U_k \subset U_{k+1}, \] and smooth maps \( g_k = 1 \) on \( U_k \)

on \( U. \)
\textbf{Step 1:} \[ \mathcal{I} = \sum_I \alpha_{I,T} \, d\tau_I \, \wedge \, d\zeta^T \] 
[\text{Here:} \{ I = p, \ldots, q \} \text{ throughout.}] 
\[ \mathcal{O} = \mathcal{J} \mathcal{X} = (-1)^p \sum_I \alpha_{I,T} \, d\tau_I \, \wedge \, d\zeta^T \quad \Rightarrow \quad \mathcal{J} \mathcal{X} = 0. \]

If we can prove \( \mathcal{J} \)-Poincaré for \( (0,2) \), then \( \mathcal{L}_I = \mathcal{J}_I \), and
\[ \mathcal{J} (\mathcal{E}^p \wedge \alpha_{I,T} \wedge \beta_I) = \sum I \alpha_{I,T} \wedge \beta_I = \alpha. \]

\textbf{Step 2:} \[ \alpha = \int d\zeta^1 \wedge \cdots \wedge d\zeta^n \}
[\text{Set} \]
\[ g(\zeta) = \frac{1}{2^n} \sum \frac{(n+1)(n+2)(\cdots)(2n-2)}{(n-1)!} \, d\zeta^1 \wedge \cdots \wedge d\zeta^n \]
\[ \mathcal{J} \mathcal{X} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial \zeta^q} = 0 \quad \text{in the plane} \]
\[ \frac{\partial g}{\partial \zeta^q} = \int d\zeta^1 \wedge \cdots \wedge d\zeta^{q-1} = \mathcal{K} \quad \text{on } U_1, \]
proving \( \mathcal{J} \)-Poincaré \textbf{locally} (i.e. on a smaller nbhd. then started with).

For \( (0,1) \) forms of a special type.

\textbf{Step 3:} \[ \alpha = \sum \alpha_{J,T} \, d\tau_J \quad \text{type } (0, 1) \quad \mathcal{J} \cdot \text{-closed (p.f. of green form use)} \]

Induce on \( k = k(\kappa) = \sup \left\{ k \mid k \in J \text{ and } \kappa \neq 0 \right\} \) (both cases \( k = q \) or step 2).

Assume done (locally) for \( k-1 \), write
\[ \alpha'' = \sum_{J \neq k} \alpha_{J,T} \, d\tau_J \]
\[ \alpha' = \sum_{J \neq k} \alpha_{J,T} \, d\tau_J \]
\[ \Rightarrow \quad \alpha = \alpha'' \wedge d\zeta_k + \alpha' \quad \text{with} \quad k(\alpha'') \geq k(\alpha) \leq k-1. \]

\[ \Rightarrow \quad 0 = \int \mathcal{X} = \frac{\partial f}{\partial \zeta^q} \wedge d\zeta_k + \int \alpha', \]
no terms with \( d\zeta_k \wedge d\zeta_k \),
\[
\Rightarrow \frac{\partial j}{\partial x} > 0 \quad \text{for } j > k \Rightarrow k > k
\]

\[
\Rightarrow \exists \, \eta \text{ s.t. } \frac{\partial j}{\partial x} = 0 \Rightarrow \frac{\partial j}{\partial x} = \delta
\]

\[
\Rightarrow \int \left( \sum_{\text{all } j} \frac{\partial j}{\partial x} \right) = (-1)^{q-1} \alpha \delta \bar{\omega} + \bar{\omega} \quad \text{where } k \leq k-1
\]

\[
\Rightarrow \alpha = \delta \beta + (z' - \bar{z}')
\]

\text{\underline{Step 4: We have proved Painlevé 3 locally for } \alpha \text{ of } (q,q) \text{ form}.}

For any \( k \), we can get \( \Psi_k^\delta \) \((q-1)\) form compactly supported on \( U_{k+1} \& \text{ct.} \). \( \delta \Psi_k = \alpha \) on \( U_k \). (Use \( u_k \) instead of \( \eta_k \).)

So why not take \( \lim_k \Psi_k \)? They might not converge uniformly on compact sets, which is necessary for \( \lim \) to commute with \( \bar{\delta} \).

It is necessary to modify the \( \Psi_k \)'s; the argument induces on \( q \).

(i) Assume the global problem (on \( U \)) solved for \( q-1 \). Then

\[
\bar{\delta} (\Psi_k^\delta - \Psi_{k+1}^\delta) = 0 \quad \text{on } U_k \Rightarrow \quad \Psi_k^\delta - \Psi_{k+1}^\delta = \bar{\delta} \beta^\delta \quad \text{on } U_k \quad \text{(truncate to } \beta := \eta_{k+1}\beta, \text{ so } \Psi_k^\delta - \Psi_{k+1}^\delta = \bar{\delta} \beta \text{ on } U_{k+1})
\]

\[
\Rightarrow \Psi_k^\delta := \Psi_{k+1}^\delta + \bar{\delta} \beta \quad \text{satisfies}
\]

\[
\bar{\delta} \Psi_k^\delta = \bar{\delta} \Psi_{k+1}^\delta = \eta \quad \text{on } U_{k+1} \& \text{agrees with } \Psi_k \text{ on } U_{k-1}
\]

(\text{truncate to } \Psi_k^\delta' = \eta_{k+1} \Psi_k \text{ on which, in addition to their properties, it vanishes appr. on } U_{k+2})

Proceed to modify the remaining \( \Psi_k \) to agree successively w/\( \Psi_{k+1} \text{ on } U_{k-2} \), done.
(ii) Since, that is, unless $q = 1$. Then $\tilde{d}(\Psi_k - \Psi_{k+1}) = 0$ on $U_k$.

$\Rightarrow$ Field morphic there.

$\Rightarrow$ If polynomial $\beta$ on $U$ approx. it to within $\frac{1}{2^k}$ on $U_{k-1}$.

$\Rightarrow \Psi_{k+1} = \Psi_k + \beta$ satisfies

$\tilde{d}\Psi_{k+1} = \tilde{d}\Psi_k = 0$ and is within $\frac{1}{2^k}$ of $\Psi_k$ on $U_{k-1}$.

$\Rightarrow \Psi = \lim \Psi_k$ exists, satisfies $\tilde{d}\Psi = 0$.

Let $E \to M$ be a holomorphic vector bundle,

(C.14) $\mathcal{A}^{0,1}(M, E) := C^\infty(M, \Lambda^* T^*(M) \otimes E)$

Since $E|_{U_k} \approx U_k \times \mathbb{C}^r$, there is a local basis of holomorphic sections

$\{ \tilde{e}_j \}_{j=1}^r$ and an arbitrary $\sigma \in C^\infty(M, E)$ is $\sigma = \sum_j f_j \tilde{e}_j$.

Set

(C.15) $$(\tilde{d}_E \sigma) := \sum_j \tilde{d}f_j \otimes \tilde{e}_j \in \mathcal{A}^{0,1}(U_k, E)$$

Over $U_{k+1}$, there is a different basis of holomorphic sections $\{ \tilde{e}_k \}_{k=1}^r$, and on $U_{k+1}$ $\tilde{e}_j = \sum_i g_{jk} \tilde{e}_k$ where the $g_{jk} \in \mathcal{O}(U_{k+1})$. Consequently $\tilde{d}$ will not depend (in $U_{k+1}$) on the choice of trivialization, and so

$$\tilde{d}_E \sigma \in \mathcal{A}^{0,1}(M, E)$$

is defined. Clearly for a given $\sigma \in \mathcal{C}^\infty(M, E)$,

(C.16) $\sigma \in \mathcal{C}^\infty(M, E) \Leftrightarrow \tilde{d}_E \sigma = 0$. 
More generally we get
\[ \partial_E : A^{0,q}(M, E) \to A^{0,q+1}(M, E) \]
and a notion of Dolbeault cohomology \( H \)

(C.7) \[
H^{q}_{\bar{\partial}}(M, E) := H^q \{ A^{0,*}(M, E), \bar{\partial}_E \}
\]
\[ \uparrow \text{complex differential} \]
\[ \text{ker } \bar{\partial}_E \cong H^{q-1}(M, E) \]
(cohomology of complex at q-th place)

which satisfies the Poincaré lemma \( H^{q}_{\bar{\partial}}(\text{polydisk}, E) = \{0\} \)
especially by Theorem C.2.

Ex/ prove a version of \( \bar{\partial} \)-Poincaré for 1-forms on a
punctured disk, using Laurent expansions (to generalize
part (ii) of Step 4 in Pf. of Thm. C.2).

Ex/ Using a covering by 2 neighborhoods \( \{ \bar{D}^1 \setminus \{0\}, \bar{D}^1 \setminus \{1\} \} \), show
that \( H^0_{\bar{\partial}}(\bar{D}^1 \setminus \{0\}, E) = \{0\} \). (You’ll again need to use
Laurent expansions at some pt., but this is independent
of the last problem.)

Ex/ Show that \( H^0_{\bar{\partial}}(C, E) \) is generated by \( \bar{\partial} \).

(Part: suppose \( \bar{\partial} = \bar{\partial} \beta \) and consider (homologous) translation
invariance of \( \bar{\partial} \).)