D. Almost complex structures

\[ M = \text{smooth manifold of dim. } m \]

**Definition 1:** (a) An almost complex structure (ACS) on \( M \) is an element \( J \in \mathcal{C}^\infty(M, \text{End}(TM)) \) s.t. \( J^2 = -\text{id} \).

(b) \( J \) is integrable \iff \( J \) structure of \( \mathbb{C} \)-manifold on \( M \) for which \( J^2 = \text{mult. by } \sqrt{-1} \) (in \( T_m p \))

**Example 1:**

- For \( \mathbb{D} = \mathbb{H} \) (Hamilton's quaternions) \( \mathbb{C} \times \mathbb{C} \) gives quaternionic structure
- For \( \mathbb{D} = \mathbb{O} \) (octonions) \( \mathbb{H} \times \mathbb{H} \) gives alternative structure

Normal division algebras \( / \mathbb{R} \)

(there are only 4)

In both cases, set \( (\alpha, \beta)^* = (\alpha^\ast, -\beta) \)

\[ (\alpha, \beta) \cdot (\gamma, \delta) = (\alpha \gamma - \delta^\ast \beta, \delta \alpha + \beta \delta^\ast) \]

inner product \( \langle p, q \rangle := \frac{1}{2} (p^* q + q^* p) \)

In both cases, set \( (\mathbb{Q}Q)^* = Q^* Q^\ast \)

Now, \( \mathbb{IR} \) = \( \{ p \in \mathbb{D}_j \mid p^* = p \} \)

\( \mathbb{II}_j \) := \( \{ p \in \mathbb{D}_j \mid p^* = -p \} \) = \( \{ \mathbb{R}^3, j^2 = 1 \} \)

\( \mathbb{II}_j \cap \mathbb{II}_j \) := \( \{ p \in \mathbb{II}_j \mid \langle p, p \rangle = 1 \} \) = \( \{ S^2, j = 1 \} \)

\( \mathbb{II}_j \) := \( \{ p \in \mathbb{II}_j \mid p^2 = -1 \} \)

\( T_p S_j = \mathbb{P}^L \cap \mathbb{II}_j \)

We claim that \( Q \mapsto QP \) defines an endomorphism \( J \) of \( T_p S_j \)
\[(QP) = P^*Q = -Q^*P = Q^*P = -QP \quad \text{if} \quad Q \in \Sigma \quad \langle P, QP \rangle = \frac{1}{2} \left( (P^*(QP) + (QP)^*P) = \frac{1}{2} \left( -PQP + PQP \right) = 0 \right) \]

(\text{If an alternating algebra}) \quad \Rightarrow \quad QP \in T_{P} \Sigma \]

Moreover, \( J^2(Q) = QP^2 = -Q \), so \( J \) defines an ACS on \( S^2 \) resp. \( S^6 \). The one on \( S^2 \) is integrable and yields the Riemann sphere \( \mathbb{C}P^1 \). As you will see (see below), the one on \( S^6 \) is NOT integrable.

**Open Problem #1**

Does there exist an integrable ACS on \( S^6 \)?

**Example 2:** On \( \mathbb{R}^2/\mathbb{Z}^2 \), there are many inequivalent ACS's. In the bases \( \mathcal{B} = \{ x^1, x^2 \} \), \( \mathcal{E} = \{ e^1, e^2 \} \)

\[
\begin{pmatrix}
J_3^z & = & \begin{pmatrix}
+1 & + (a^2 + b^2) \\
-1 & -a/b
\end{pmatrix} \\
J_3^x & = & \begin{pmatrix}
\frac{a}{b} & -1 \\
\frac{a^2 + b^2}{b} & -a/b
\end{pmatrix}
\end{pmatrix}
\]

yields an integrable ACS (complex structure) with holomorphic vector field resp. differential 1-form \( \omega \) for the \( (\cdot i) \) resp. \( (-i) \) eigenvectors of \( J \) resp. \( J^\perp : \text{e.g.} \)

\[ \omega = dx + (a + bi) dy \quad (=: dx + idy) \]

Integrating along \( \partial \mathbb{R}^2 \), we see this as a C-manifold, with this C-str.

\[ (\mathbb{R}^2/\mathbb{Z}^2, J) \cong (\mathbb{C}/\mathbb{Z} \langle 1, a + bi \rangle, \omega) \]

\* i.e. the structure of a C-manifold on \( \mathbb{R}^2/\mathbb{Z}^2 \)
Definition 2: (i) A (smooth) distribution \( \mathcal{D} \) on \( M \) is a smooth (rank \( n \)) vector subbundle \( \mathcal{D} \subset T_M \).

(ii) A vector field is said to lie in \( \mathcal{D} \) ("\( \mathcal{D} \) \) if it is a section of \( \mathcal{D} \).

(iii) An integral manifold of \( \mathcal{D} \) is a submanifold \( N \subset M \) s.t.

\[
T_N = \mathcal{D} \mid_{T_N} \quad \text{resp.} \quad T_N \subset \mathcal{D}.
\]

(iv) \( \mathcal{D} \) is involutive \( \iff \) \([\mathcal{D}, \mathcal{D}] \subset \mathcal{D} \), i.e., the Lie bracket of any two vector fields in \( \mathcal{D} \) again \( \in \mathcal{D} \).

(v) \( \mathcal{D} \) is integrable \( \iff \) locally \( M \) has a regular foliation by integral manifolds of \( \mathcal{D} \), i.e., for some open cover \( \{U_k\} \) there are \( C^\infty \) submersions \( \pi_k : U_k \rightarrow \mathbb{R}^m \) s.t.

\[
\forall p \in U_k \quad \ker(\pi_k \mid_p) = \mathcal{D}_p.
\]

We shall take for granted the following result from differential topology (which geometry I is supposed to cover, and a complete proof/discussion of which may be found on pp. 42-46 of [WARNER]):

Frobenius Theorem: \( (iv) \iff (v) \).

(The hard direction is "\( \Rightarrow \)".)
Now, we can re-do Def. 2 for $M = \text{C-manifold of dim } n, \mathcal{D} = \text{holomorphic distribution (algebraic def.) of rank k}$. If the definition of involutivity is the same, and holomorphic integrability means that $\phi_k$ is holomorphic, then:

**Theorem 1**: \(\mathcal{D} \text{ involutive } \iff \mathcal{D} \text{ holomorphically integrable}.

**Proof**: Again, is trivial.

(\(\Rightarrow\)): Frobenius theorem gives smooth submersion $\phi : U \to \mathbb{R}^{2n-2k}$ with $\ker \phi_k|_p = \mathcal{D}_p$.

Consider holomorphic submanifolds $\mathcal{W}_U \subset U$ transverse to $\mathcal{D}_p : T_p U \oplus \mathcal{D}_p = T_p U$.

And let $\chi^j$ be basis of hol. sections of $\mathcal{D}$ given by $\left(\frac{\partial}{\partial x^j}\right)_p$. Let $\phi^x = \text{restriction of } \phi \text{ to } \mathcal{W}_U$.

2 facts: (a) given any vector field $\theta$ in $U$, holomorphically

\[ \exists \chi^j \Rightarrow [\chi^j, J\theta] = J[\chi^j, \theta]. \]

(b) $\frac{\partial}{\partial x^j}\phi^x \theta = \phi^x [\chi^j, \theta]$.

Bring these facts to bear upon the unique lift $\tilde{\omega}$ to $U$.

\[
\text{and (2)} \text{ which can assume these norms } (2, 1, \ldots, 2k) = (e_1, \ldots, e_k).
\]
(of a $C^\infty$ vector field $\nu$ on $\mathbb{R}^{2n-2h}$) which lies in the transverse distribution (defined by the $N$),

$$0 = \frac{d}{d\tau} \nu \Rightarrow \left[ X^j, \nu \right] \in \mathcal{D} \Rightarrow \left[ X^j, J\nu \right] \in \mathcal{D}$$

(a) + (b) $\Rightarrow \theta^* J\nu = 0$, which means that $\theta^* J^*\theta$ gives an ACS on $\mathbb{R}^{2n-2h}$ independent of $t$.

We can now use the $N$ to define a complex structure on $\mathbb{R}^{2n-2h}$. “Compatibility of ACS for different $t$” ensures compatibility of these complex structures (i.e., $C$-linearity / solving C-R eqns at each pt. $\Rightarrow$ holomorphy). Likewise, $\theta$-closedness of $\ker(\theta^*_k) = \Gamma$-linearity of $\theta^*_k$, $\theta^*_k \Rightarrow$ holomorphy of $\theta$.

Remark: it’s a bit more complicated than Vojin’s lets on.

Given an ACS on a smooth manifold $M$, we can define (smooth) subbundles $T^1,\mathbb{C}_M$, $T^0,\mathbb{C}_M$ of $T^\mathbb{C}_M$ by the $(\pm i)$ -eigen spaces of $J$ on $T_M$.

* Note: we could have worked with $C^1$ manifolds/distributions/ACS; N-NN still holds.
Theorem 2:
(Weaver-Mironescu)

\[ J \text{ is integrable} \implies \text{either (hence both) of } T^{1,0}_m / T^{0,1}_m \text{ is involutive.} \]

Before the (partial) proof, an appellation. \( (\forall m \implies \phi_{\bar{m}}) \)

Example 3: \( M = \) smooth oriented real surface

Prop.: (a) Every ACS on \( M \) is integrable

(b) There is a bijection

\[ \{ \text{ACS on } M \} \leftrightarrow \{ \text{Riemannian metric on } M \} \]

Proof: (a) Since \( T^{1,0}_m \) has rank 1, any 2 sections are proportional by a \( C^\infty \) function \( f \):

\[ [X, fX] = f [X, X] = X(f) X \implies [T^{1,0}_m, T^{1,0}_m] \subset T^{1,0}_m \]

(b) Given nowhere vanishing \( g \in C^\infty (M, (\mathbb{S}^2 T_m)^* ) \), determine \( J \) uniquely by the rule:

For each \( \nu \in T_{m,p} \), with \( g(\nu, \nu) = 1 \):

\[ \{ \nu, J \nu \} \text{ is a positively oriented } \]

\( \text{orthonormal basis} \)

Conversely, given \( J, \nu \in T_{m,p} \), define \( g_p, g \) to scale by

\[ g_p (J \nu, J \nu) = g_p (\nu, \nu) \]

Remaining details are left to you.
The main theme is that (on $M$) any Riemannian metric defines a complex manifold structure.

Proof of Thm. 2 for $M$ of class $C^\omega$:

It suffices to do this locally, i.e., on $U \times \mathbb{C}^n = \mathbb{R}^{2n}.$

Step 1: Extend $T^{0,1}$ to a holomorphic distribution on a larger space:

$\sum E_{\bar{z}} x^i = \mathcal{J}_x \in C^\omega (V_x, \text{End} (\mathbb{R}^{2n}))$

$\sum E_{\bar{z}} \bar{z}^i = \mathcal{J}_x \in C^\omega (V_x, \text{End} (\mathbb{C}^{2n}))$

So we have $\partial / \partial z_j$ (on $\mathbb{R}^{2n}$, trans.),

$C \langle \{ \delta_j \} \rangle = T^{0,1} V = \bigcup_{V \ni \mathcal{C}} T^{0,1} V$

$C \langle \{ \delta_j \} \rangle = C \langle \{ \delta_j \} \rangle$

and $\mathcal{J}_V$ is our extension.

Step 2: This extension is integrable:

$[\delta_j + i \delta_j, \delta_k + i \delta_k] = \sum \alpha_k \delta_j \delta_k$ $+$ $\sum \beta_j \delta_k \delta_j$

where $\beta_j \delta_j$ vanish identically along $\mathcal{V} = \mathcal{0}$ on $V$.

Now hold Frobenius $\Rightarrow \mathcal{J} \beta : \mathcal{V} \rightarrow \mathbb{C}^n$, holomorphic submersion with $T(\text{fiber}) = \mathcal{V}$.
Step 3: $\phi^*_V$ is a local diffeomorphism:

$\mathbb{R}^n, C^n \in C^n$ are "transverse" in above diagram $\Rightarrow$

$(E_V^1)_V \oplus T_V = (T_V^\perp)_V^\mathbb{R}$. So the fibers of $\phi$ are transverse to $V$, done (by holo. inverse function theorem).

Step 4: This local diffeo. respects the ACS:

$T_V \hookrightarrow T^\perp_V \rightarrow (T^\perp_V/E_V)_V$ is this composition $C$-linear?

$j_j \quad \mapsto \quad \overline{j_j} \quad \rightarrow \quad [\overline{j_j}]$

$J j_j \quad \mapsto \quad J \overline{j_j} \quad \rightarrow \quad [\overline{J j_j}] = [\overline{J} \overline{j_j}] \quad \Rightarrow \quad \text{yes.}$

(We see $E_V \rightarrow \overline{j_j} + i J \overline{j_j}$)

We conclude that $(V, J) \xrightarrow{\phi^*_V} (C^n, J^{\phi^*_V})$ exhibits $J$ as integrable, done.

Ex) If $M$ is an ACS then $M$ is orientable.

* for $n = 1$, we are saying that

$\mathbb{R} \langle x + i y, (x + i y)^2 \rangle \cong \mathbb{R} \langle x, y, z_1, z_2, i z_1, i z_2 \rangle$
Ex / $M$ $C^\infty$ manifold, $\mathcal{J} \text{ ACS}, \ X, Y \in C^\infty(M, T_m)$

Define Nijenhuis tensor of $\mathcal{J}$ by

$-N(X,Y) := J[JX, JY] + J[JY, X] + [X, Y] - [JX, JY]$

(a) Show $N$ bilinear over $C^\infty(M)$, antisymmetric, hence gives a section in $C^\infty(M, \text{Hom}(\Lambda^2 T_m, T_m))$.

(b) Show $\mathcal{J}$ integrable $\iff$ $N = 0$.

Ex / Show the ACS (in Example 0.1 above) on $S^6$ is not integrable.