D. Almost complex structures

M = smooth manifold of dim. m (even)

Definition 1: (a) An almost complex structure (A.C.S.) on M is an element $J \in C^\infty(M, \mathrm{End}(T_M))$ s.t. $J^2 = -\mathrm{id}$.

(b) J is integrable $\iff$ $\exists$ structure of $\mathbb{C}$-manifold on M for which J^2 = mult. by $\sqrt{-1}$ (on $T_{M,p}$)

Example 1: $\begin{cases} \mathbb{D}_1 = \mathbb{H} \text{ (Hamilton's quaternions) } \overset{\text{set}}{=} \mathbb{C}\times\mathbb{C} & \text{associative} \\ \mathbb{D}_2 = \mathbb{O} \text{ (octonions) } \overset{\text{set}}{=} \mathbb{H}\times\mathbb{H} & \text{alternative} \end{cases}$

normed division algebras $/\mathbb{R}$ (there are only 4)

Subalgebra gen. by any 2 elts. is associative

In both cases, set $(\alpha,\beta)^* = (\alpha^*, -\beta)$

$$(\alpha,\beta)\cdot(\gamma,\delta) = (\alpha\gamma - \delta^*\beta, \delta\alpha + \beta\gamma^*)$$

inner product $\rightsquigarrow$ $\langle P, Q\rangle := \frac{1}{2}(P^*Q + Q^*P)$ $\hookleftarrow$ restricted to $\mathbb{I}_j$ below, it's just a Euclidean dot product

one can check that $(PQ)^* = Q^*P^*$

Now, $\mathbb{R} = \{P\in\mathbb{D}_j \mid P^* = P\}$

$$\mathbb{I}_j := \{P\in \mathbb{D}_j \mid P^* = -P\} \cong \begin{cases}\mathbb{R}^3, & j=1\\ \mathbb{R}^7, & j=2\end{cases}$$

$$\mathbb{I}_j \supset \mathbb{S}_j := \{P\in\mathbb{I}_j \mid \underbrace{\langle P,P\rangle = 1}_{\text{i.e. } P^2=-1}\} = \begin{cases} S^2, & j=1 \\ S^6, & j=2\end{cases}$$

$$T_P\mathbb{S}_j = P^\perp \cap \mathbb{I}_j$$

We claim that $Q \mapsto QP$ defines an endomorphism J of $T_P\mathbb{S}_j$:

$$(QP)^* = P^*Q^* \underset{\uparrow\atop Q\in\mathfrak{I}_j}{=} -P^*Q \underset{\uparrow\atop \langle P,Q\rangle=0}{=} Q^*P \underset{\uparrow\atop P\in\mathfrak{I}_j}{=} -QP$$

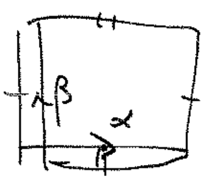
$$\langle P, QP\rangle = \tfrac{1}{2}\left(P^*(QP) + (QP)^*P\right) \underset{\uparrow}{=} \tfrac{1}{2}\left(-PQP + PQP\right) = 0$$

(① on alternative algebra) $\Rightarrow QP \in T_P\mathfrak{S}_j$

Moreover, $J^2(Q) = QP^2 = -Q$, so J defines an ACS on S^2 resp. S^6. The one on S^2 is integrable and yields the Riemann sphere $\mathbb{C}P^1$. As you will show (see below), the one on S^6 is NOT integrable. □

$\boxed{\text{Open problem \# 1}}$ Does there exist an integrable ACS on S^6?

Example 2: On $\mathbb{R}^2/\mathbb{Z}^2$ (coords (X,Y)), there are many inequivalent ACS's. In the bases $\underline{\partial} = \{\partial/\partial X, \partial/\partial Y\}$, $\underline{\partial}^\vee = \{dX, dY\}$



$$[J]_{\underline{\partial}} = \begin{pmatrix} +a/b & +\frac{(a^2+b^2)}{b} \\ -1/b & -a/b \end{pmatrix}, \quad [J^\vee]_{\underline{\partial}^\vee} = {}^t[J] = \begin{pmatrix} a/b & -1/b \\ \frac{a^2+b^2}{b} & -a/b \end{pmatrix}$$

yields an integrable ACS ($\Leftrightarrow$ complex structure*) with holomorphic vector field resp. differential 1-form the $(+i)$ resp $(-i)$ eigenvectors of J resp. $J^\vee$: e.g.,

$$dz = dX + (a+bi)\,dY \quad (=: dx + i\,dy)$$

Integrating along α & β, we see that as a $\mathbb{C}$-manifold, with this $\mathbb{C}$-str.,

$$(\mathbb{R}^2/\mathbb{Z}^2, J) \cong \left(\boxed{\mathbb{C}/\mathbb{Z}\langle 1, a+bi\rangle}, \mu_i\right),$$

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* i.e. the structure of a $\mathbb{C}$-manifold on $\mathbb{R}^2/\mathbb{Z}^2$

<u>Definition 2</u>: (i) A (rank r) <u>distribution $\mathcal{D}$ on M</u> is a smooth (rank r) vector subbundle $\mathcal{D} \subset T_M$.

(ii) A vector field is said to <u>lie in $\mathcal{D}$</u> ("$\xi \in \mathcal{D}$") if it is a section of $\mathcal{D}$

(iii) An $\left.\begin{matrix}\text{integral manifold} \\ \text{resp. submanifold}\end{matrix}\right\}$ of $\mathcal{D}$ is a submanifold

$N \subset M$ s.t. $\left.\begin{matrix} T_N = \mathcal{D} \\ \text{resp. } T_N \subset \mathcal{D}\end{matrix}\right\}$.

(iv) $\mathcal{D}$ is <u>involutive</u> $\iff [\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$, i.e. the Lie brackets of any 2 vector fields in $\mathcal{D}$ is again $\in \mathcal{D}$

(v) $\mathcal{D}$ is <u>integrable</u> $\iff$ locally M has a regular foliation by integral manifolds of $\mathcal{D}$, i.e. (for some open cover $\{\mathcal{U}_\alpha\}$) there are C^∞ submersions

$$\phi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^{m-r}$$

s.t. $\forall p \in \mathcal{U}_\alpha$) $\underbrace{\ker(\phi_{\alpha *}|_p)}$ $= \mathcal{D}_p$

(i.e. the tangent space to the leaf $\phi_\alpha^{-1}(\phi_\alpha(p))$ of the foliation thru p)

We shall take for granted the following result from differential topology (which Geometry I is supposed to cover, <u>and</u> a complete proof/discussion of which may be found on pp. 42–46 of [WARNER]):

<u>Frobenius Theorem</u>: (iv) $\iff$ (v).

(The hard direction is "$\Rightarrow$")

Now, we can re-do Defn 2 for M = $\underline{\mathbb{C}\text{-manifold of}}$ $\underline{\dim. n}$, $\mathcal{D}$ = $\underline{\text{holomorphic distribution}}$ (obvious defn.) $\underline{\text{of rank } k}$;

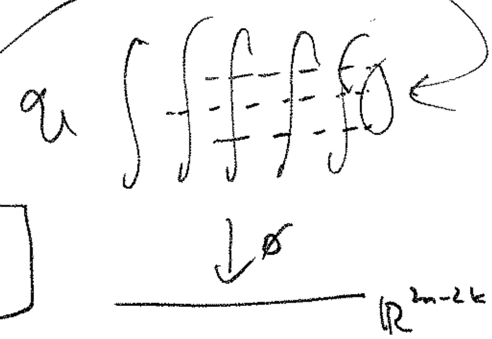
the definition of involutivity is the same, and holomorphic integrability means that ϕ_α is holomorphic ($\underset{\text{rank thm.}}{\Longleftrightarrow}$ integral manifolds are $\mathbb{C}$-manifolds).

$\underline{\text{Theorem 1}}$: In this case, $\mathcal{D}$ involutive $\Longleftrightarrow$ $\mathcal{D}$ holomorphically integrable.
(Holomorphic Frobenius)

$\underline{\text{Proof}}$: Again, "$\Leftarrow$" is trivial.

$$\begin{array}{c}\mathbb{R}^{2n}_{(x_1,\ldots,x_{2n})}\\ \shortparallel\\ \mathbb{C}^n_{(z_1,\ldots,z_n)}\\ \cup\end{array}$$

($\Rightarrow$): Frobenius thm. gives $\underline{\text{smooth}}$ submersions $\phi : \mathcal{U} \to \mathbb{R}^{2n-2k}$ with $\ker \phi_*|_p = \mathcal{D}_p$.



Consider holo. $(n-k)$ submanifolds* $W_{(t_1,\ldots,t_k)} \subset \mathcal{U}$ transverse to $\mathcal{D}_p$: $\boxed{T_p W_{\underline{t}} \oplus \mathcal{D}_p = T_p \mathcal{U}}$

and let χ^j = basis of holo. sections of $\mathcal{D}$ given by (projection to $\mathcal{D}_p$ of) $\{\partial/\partial z_j\}_{j=1}^k$

$\phi^{\underline{t}}$ = restriction of ϕ to $W_{\underline{t}}$

$\underline{\text{2 facts}}$: (a) given any vector field θ on $\mathcal{U}$, holomorphicity of χ_j $\Rightarrow$ $[\chi^j, J\theta] = J[\chi^j, \theta]$.

(b) $\partial/\partial t_j \, \phi^t_* \theta = \phi^t_* [\chi^j, \theta]$.

Bringing these facts to bear upon the unique lift $\tilde{v}$ to $\mathcal{U}$

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* wolog can assume this means $(z_1,\ldots,z_k) = (t_1,\ldots,t_k)$

[illegible] of

(of a C^∞ vector field v on $\mathbb{R}^{2n-2k}$) which lies in the transverse distribution (defined by the $W_{\underline{t}}$'s),

$$0 = \frac{\partial}{\partial t_j} 0 = \frac{\partial}{\partial t_j} \phi_* \tilde{v} \underset{(b)}{\Longrightarrow} [X^j, \tilde{v}] \in \mathcal{D} \underset{\left(\substack{(a) + \\ \text{holomorphicity} \\ \text{of } \mathcal{D}\ (\Leftrightarrow J\mathcal{D} = \mathcal{D})}\right)}{\Longrightarrow} [X^j, J\tilde{v}] \in \mathcal{D}$$

$$\underset{(b)}{\Longrightarrow} \frac{\partial}{\partial t_j} \phi_*^{\pm} J\tilde{v} = 0 ,$$ which means that $\phi_*^{\pm} \circ J \circ (\phi_*^{\pm})^{-1}$ gives an ACS on $\mathbb{R}^{2n-2k}$ independent of $\underline{t}$.

We can now use the $W_{\underline{t}}$ to define a complex structure on $\mathbb{R}^{2n-2k}$. "Compatibility of ACS for different $\underline{t}$" ensures compatibility of these complex structures (i.e. $\mathbb{C}$-linearity / solving C-R eqns. at each pt. $\Rightarrow$ holomorphicity). Likewise, $\mathcal{O}$-closedness of $\ker(\phi_*)$ $\Rightarrow$ $\mathbb{C}$-linearity of $\phi_*|_p$'s $\Rightarrow$ holomorphicity of ϕ.

$\square$

Remark: it's a bit more complicated than Voisin lets on.

Given an ACS on a smooth* manifold M, we can define (smooth complex) subbundles $T_M^{1,0}$, $T_M^{0,1}$ of $T_M \otimes_{\mathbb{R}} \mathbb{C}$ by the $(\pm i)$-eigenspaces of J on T_M.

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* Note: we could have worked with C^1 manifold / distribution / ACS; N-N still holds

Theorem 2 (Newlander–Nirenberg 1957): J is integrable $\Longleftrightarrow$ either (hence both) of $T_M^{1,0}/T_M^{0,1}$ is involutive.

Before the (partial) proof, an application. (Note: $\Rightarrow$ easy.)

Example 3: $M =$ smooth (or C^1) oriented real surface

Prop.: (a) Every ACS on M is integrable

(b) There is a bijection

$$\{\text{ACS on } M\} \longleftrightarrow \left\{ \frac{\text{Riemannian metrics on } M}{\text{Conformal} \equiv} \right\}$$

Pf.: (a) Since $T_M^{1,0}$ has rank 1, any 2 sections are proportional by a C^∞ function f:

$[X, fX] = f[X,X] = X(f)X \implies [T^{1,0}, T^{1,0}] \subset T^{1,0}$

$\underset{NN}{\implies}$ done.

(b) Given nowhere vanishing $g \in C^\infty(M, (S^2 T_M)^\vee)$, determine J uniquely by the rule:

For each $v \in T_{M,p}$ with $g(v,v) = 1$, $\{v, Jv\}$ is a positively oriented orthonormal basis.

Conversely, given J, $v \in T_{M,p}$, define g_p up to scale by

$$g_p(Jv, Jv) = g_p(v,v).$$

Remaining details are left to you. $\square$

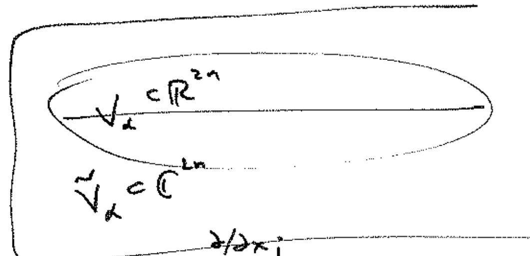
The take-home is that (on M) any Riemannian metric defines a complex manifold structure.

Proof of Thm. 2 ($\Leftarrow$) for M of class C^ω (real analytic transition fcns.): It suffices to do this locally, i.e. on $\mathcal{U}_\alpha \xrightarrow[\phi_\alpha]{\cong} V_\alpha \subset \mathbb{R}^{2n}$.

Step 1 Extend $T^{0,1}$ to a holo. distribution on a larger space:

$$\sum \varepsilon_I \underline{x}^I = J_\alpha \in C^\omega(V_\alpha, \operatorname{End}(\mathbb{R}^{2n}))$$ extends to

$$\sum \varepsilon_I \underline{z}^I = \tilde{J}_\alpha \in \mathcal{O}(\tilde{V}_\alpha) \otimes \operatorname{End}(\mathbb{C}^{2n})$$



So we have

$$\mathbb{C}\langle\{\tilde{\partial}_j\}_{j=1}^{2n}\rangle = T\tilde{V}|_V \cong TV \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}\langle\{\partial_j\}_{j=1}^{2n}\rangle$$

$\tilde{\partial}_j = \partial/\partial z_j$ (on holo. fcns), $\partial_j = \partial/\partial x_j$

$$\cup \qquad\qquad \cup$$

$$\mathbb{C}\langle\{\tilde{\partial}_j + i\tilde{J}\tilde{\partial}_j\}_{j=1}^{n}\rangle =: E_{\tilde{V}}|_V \cong T^{0,1}_V = \mathbb{C}\langle\{\partial_j + iJ\partial_j\}_{j=1}^{n}\rangle$$

$\mathbb{R}^{2n} \subset \mathbb{C}^{2n} \supset \mathbb{C}^n$

and $E_{\tilde{V}}$ ($= (-i)$-eigenspace of $\tilde{J}$) is our extension.

Step 2 This extension is integrable:

$$[\tilde{\partial}_j + i\tilde{J}\tilde{\partial}_j, \tilde{\partial}_k + i\tilde{J}\tilde{\partial}_k] = \sum \alpha^{\ell}_{jk}(z)\{\tilde{\partial}_\ell + i\tilde{J}\tilde{\partial}_\ell\} + \sum \beta^{\lambda}_{jk}(z)\{\tilde{\partial}_\lambda - i\tilde{J}\tilde{\partial}_\lambda\}$$

(holo.)

write as holo. section of $T\tilde{V}$

where β^{λ}_{jk} vanish identically along $V \implies \equiv 0$ on $\tilde{V}$.

NEW

holo. Frobenius $\implies \exists\, \phi : \tilde{V} \to \mathbb{C}^n$ holo. submersion with $T(\text{fibers}) = E_{\tilde{V}}$.

Step 3 $\phi|_V$ is a local diffeomorphism :

$\mathbb{R}^{2n}, \mathbb{C}^n \subset \mathbb{C}^{2n}$ are "transverse" in above diagram* $\Rightarrow$

$\left(E_{\tilde{V}}|_V\right)^{\mathbb{R}} \oplus T_V = \left(T_{\tilde{V}}|_V\right)^{\mathbb{R}}$. So the fibers of ϕ are transverse

to V, done (by holo. inverse function theorem).

Step 4 This local diffeo. respects the ACS :

$$\begin{tikzcd} T_V \arrow[r, hook] \arrow[rr, bend left, "(\phi|_V)_*"] & T_{\tilde{V}}|_V \arrow[r] & (T_{\tilde{V}}/E_{\tilde{V}})|_V \end{tikzcd}$$

is this composition $\mathbb{C}$-linear ?

$J \circlearrowleft \quad\quad\quad\quad \circlearrowleft \mu_{\sqrt{-1}}$

$$\partial_j \longmapsto \tilde{\partial}_j \longmapsto [\tilde{\partial}_j]$$

$$J\partial_j \longmapsto \tilde{J}\tilde{\partial}_j \longmapsto [\tilde{J}\tilde{\partial}_j] \equiv [\sqrt{-1}\,\tilde{\partial}_j] \quad \Longrightarrow \text{ YES.}$$

(since $E_{\tilde{V}} \ni \tilde{\partial}_j + i\tilde{J}\tilde{\partial}_j$)

We conclude that $(V, J) \xrightarrow[\phi|_V]{\cong} (\mathbb{C}^n, \mu_{\sqrt{-1}})$ exhibits J

as integrable, done. $\square$

Ex/ If M has an ACS then M is orientable.

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* for $n=1$, we are saying that

$$\mathbb{R}\langle \partial_1 + i\partial_2,\ i(\partial_1 + i\partial_2)\rangle \oplus \mathbb{R}\langle \partial_1, \partial_2\rangle = \overbrace{\mathbb{R}\langle \partial_1, i\partial_1, \partial_2, i\partial_2\rangle}^{\mathbb{C}\langle \partial_1, \partial_2\rangle}$$

Ex/ M C^∞ mfld., J ACS, $X, Y \in C^\infty(M, T_M)$

Define Nijenhuis torsion of J by

$$-N(X,Y) := J[X, JY] + J[JX, Y] + [X,Y] - [JX, JY]$$

(a) Show N bilinear over $C^\infty(M)$, antisymmetric, hence gives a section in $C^\infty(M, \mathrm{Hom}(\Lambda^2 T_M, T_M))$.

(b) Show J integrable $\iff$ $N = 0$.

Ex/ Show the ACS (in Example D.1 above) on S^6 is not integrable.