E. Kähler manifolds

A Riemannian metric on a smooth manifold $M$ is an everywhere positive-definite section $g \in C^\infty(M, Sym^2 T^*_M)$; the pair $(M, g)$ is called a Riemannian manifold.

In the surface case, this was (up to scaling) essentially the same as an A.C.S.; in general, while this isn't so, it remains of interest to look at Riemannian metrics compatible with an A.C.S. This leads to the notion of a Hermitian manifold and, after imposing another condition, a Kähler manifold. The first time you see it, the Riemannian-Hermitian business can be confusing, so it's best to do it first "on a single tangent plane".

More Linear Algebra

Let $V = \mathbb{R}$-vector space with basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}

W := V^* = \mathbb{R}$-vector space with basis $\{dx_1, \ldots, dx_n, dy_1, \ldots, dy_n\}$

we have $(W, J) \cong (V, \Omega)$.

Write $\{dx_j, dy_j\}$ for the dual bases; as usual, for $W$

resp. $W^*$ we have the bases $\{\frac{\partial}{\partial y_j}, \frac{\partial}{\partial x_j}\}$ resp. $\{dx_j, dy_j\}$. 
Theorem 1: TFAE:

(i) a (real) symmetric bilinear form \( g \) on \( W \) compatible with \( J \)

(ii) a (real) alternating bilinear form \( h \) on \( W \) compatible with \( J \)

(iii) an Hermitian form \( h \) on \( V \)

Also, they are all non-degenerate (or not) together; and (i) and (iii)
are positive-definite (or not) together, in which case we also
write "\( \omega > 0 \)."

Proof: Start with (iii); since \( h \) is linear and \( h(u,v) = h(v,u) \),

\[ h = \sum h_{ijk} \partial x_i \otimes \partial x_j + \sum h_{ij} \partial y_j \otimes \partial x_i \]

where \( h_{ijk} = h_{jik} \)

Extending this to \((W^\vee)^\otimes 2\),

\[ h = \sum h_{ijk} \partial y_i \otimes \partial y_k \]

so

\[ g := \text{Re } h = \frac{1}{2} \left\{ \sum h_{ijk} \partial x_j \otimes \partial x_k + \sum \overline{h_{ijk}} \partial y_j \otimes \partial y_k \right\} = \frac{1}{2} \sum h_{ijk} \partial y_j \partial x_k \]

\[ \omega := -i \text{Im } h = \frac{i}{2} \left\{ \sum h_{ijk} \partial x_j \partial x_k - \sum \overline{h_{ijk}} \partial x_j \partial y_k \right\} \]

(E.1) \[ \frac{i}{2} \sum h_{ijk} \partial x_j \partial x_k \in \Lambda^2 W^\vee \cap \Lambda^1 W_\omega^\vee \]

Here \( \partial x_j \partial x_k = \partial x_j \partial x_k + \partial x_k \partial x_j \),

and we can say \( g \) resp. as \( \omega \) resp. in \( \text{Sym}^2 W^\vee \) resp. \( \Lambda^2 W^\vee \) (resp. \( W_\omega^\vee \))

since (by construction) they are real.
Assume $h > 0$ (i.e. $h(u, v) > 0$ $\forall u, v \in W_{(0)}$).

By Gram-Schmidt, there is a unitary basis, in terms of which

\[
\begin{align*}
 h &= \Sigma \langle x, \bar{x} \rangle, \quad \omega = \frac{i}{2} \Sigma \langle x, x \rangle, \\
 g &= \frac{i}{2} \Sigma \langle x, \bar{x} \rangle = \frac{i}{2} \Sigma (\langle x_j, \bar{x}_j \rangle + \langle \bar{x}_j, x_j \rangle) (\geq 0).
\end{align*}
\]

Another way to look at all this is: with respect to the real
basis $e'$ we have, writing $H = \{ h_{j,k} \} = S - iA$ where $S := Re H, \\
A := -Im H$, and $H = \bar{H}$, $S = S^T$, $A = -A$,

\[
\begin{align*}
 [h]_{e'} &= \begin{pmatrix} H & -iA \\ iA & \bar{H} \end{pmatrix}, \quad [g]_{e'} = \begin{pmatrix} S & A \\ -A & S \end{pmatrix}, \quad [\omega]_{e'} = \begin{pmatrix} i & S \\ -S & i \end{pmatrix}
\end{align*}
\]

which w.r.t. $e''$ (non-real, hence can't just take $\text{Re/Im of matrices}$)

\[
\begin{align*}
 [h]_{e''} &= \begin{pmatrix} 0 & \bar{H} \\ \bar{H} & 0 \end{pmatrix}, \quad [g]_{e''} = \frac{i}{2} \begin{pmatrix} 0 & H \\ -\bar{H} & 0 \end{pmatrix}, \quad [\omega]_{e''} = \frac{i}{2} \begin{pmatrix} 0 & \bar{H} \\ -H & 0 \end{pmatrix},
\end{align*}
\]

either way, it's clear that $h = g - i\omega$, and $[\bar{J}] [g] [J] = [g]$ \\
$[\bar{J}] [\omega] [J] = [\omega]$. \\

Going back the other way, let $g : W \times W \to \mathbb{R}$ be a symmetric bilinear form with the
compatibility condition $g(Ju, Jv) = g(u, v)$. Then

$\omega(u, v) = g(Ju, v) = g(J^2u, Jv) = -g(u, Jv) = g(Jv, u) = -\omega(v, u)
$

is antisymmetric, i.e. $\Lambda W^\vee$. Noting that $J$ acts on

$\Lambda^p W^\vee$ by $\bar{J}^p = J^p$, and

\[
\Lambda^2 W^\vee = \left( \Lambda^2 W^\vee \oplus \Lambda^0 W^\vee \right) \oplus \Lambda^1 W^\vee
\]

the $(-1)$-eigenspace $\Lambda^1 W^\vee$ we have

\[
\bar{J}^2 = -1
\]
\( \omega(Jv, Jv) = g(J^2 v, Jv) = -g(v, v) = \omega(v, v) \)
\( \Rightarrow \omega \in \Lambda^{1,1} W^\circ \), and we also see the equivalence of this condition and \( \omega \) is \( J \)-invariant. (One could also start with \( \omega \in \Lambda^{2,0} W^\circ \cap \Lambda^{0,2} W^\circ \) and set \( g(v, v) = \omega(v, Jv) \).)
Taking \( h = g - i\omega \) gives a Hermitian form, finishing the job.

**Corollary 1:** If \((M, J)\) is a complex manifold, TFAE:

(i) a Riemannian metric \( g \) compatible with \( J \)

(ii) a positive real \((1,1)\)-form \( \omega \in A^2(M) \cap A^{1,1}(M) =: A^{1,1}_g(M) \)

(iii) a \((C^\infty)\) Hermitian metric \( h \) on \( \overline{TM} \).

**Definition 1:** Let \( M \) be a complex manifold with \( \omega \in A^{1,1}_g(M) \)
\( \omega > 0. \) \( M \) is **Kähler** \( \iff \Delta\omega = 0 \). (\( \omega \) is called the **Kähler form**, and \( g(\cdot, h) \) the **Kähler metric**.)

**Definition 2:** A **symplectic manifold** is a smooth \( 2n \)-manifold \( M \) equipped with a nondegenerate form \( \omega \in A^2(M) \) \( \omega^n := \frac{\omega \wedge \ldots \wedge \omega}{n!} \) is nowhere zero.

**Corollary 2:** Every Kähler manifold is symplectic; in fact,

\[
\frac{\omega^n}{h^\frac{n}{2}} = d\text{vol}(g). \]
Proof: In the unitary basis at a point $p$,
\[
\omega_p = \frac{i}{2} \sum \delta x_i \wedge \delta y_j = \frac{i}{2} \delta x_j \wedge \delta y_j.
\]
\[
\omega^n_p = n! \delta x_1 \wedge \delta y_1 \wedge \cdots \wedge \delta x_n \wedge \delta y_n = n! \text{vol}(\mathbb{C}_j(\delta x_j + i\delta y_j)).
\]
\[
\square
\]

Corollary 3: Let $M$ be compact Kähler if $\dim(M) = n$.

Proof: Obviously $\omega = 0 \implies \omega^n = 0$.

Suppose $\omega^k = d\omega$, $\omega \in A^{2k}(\mathbb{R})$. Then
\[
\omega^n = d(\omega^{n-k} \wedge \omega) \implies \int_M \omega^n = 0
\]
\[
\square
\]

Corollary 4:
(i) Let $N \subset M$ be a complex submanifold of a Kähler manifold $M$. Then $N$ is also Kähler.

(ii) Wirtinger's Theorem $N$ compact of dim. $d$.

(E.4)
\[
\text{vol}(N) = \frac{1}{d!} \int_N \omega \wedge d\omega
\]

Proof: $i^*g$ gives a Riemannian metric on $N$ (clearly $> 0$).

By compatibility of $J$'s, it is clear that $i^*\omega \in A^{2k}(N)$ is the (1,1) form associated to $i^*g$, and hence is $> 0 \implies N$ Kähler.

Done by Corollary 2.

\[
\square
\]

\*\* i.e. $N$ is the image of a holomorphic immersion of a Cauchy domain.

\*\* Equiv. locally cut out by holomorphic coordinate.

\*\* $T_p N$ everywhere closed under $J$.

\*\* Use Neuenhauer-Nirenberg

\*\* Note: (E.2) can only be arranged, by choice of a holomorphic coordinate system, at a single point — NOT on the whole neighborhood.
Definition 3: The singular homology of a smooth manifold $M$ is the homology of the complex of singular chains

$$\cdots \to C_{q+1}(M; \mathbb{Z}) \xrightarrow{\partial} C_q(M; \mathbb{Z}) \xrightarrow{\partial} C_{q-1}(M; \mathbb{Z}) \to \cdots$$

(E.5) $H_q(M; \mathbb{Z}) := \frac{\ker \delta}{\text{im} \delta} = \frac{\text{"cycles"}}{\text{"boundaries"}}$.

Here $C_q(M; \mathbb{Z}) := \mathbb{Z} \langle C^0(\Delta^q, M) \rangle$, and

$$\delta^q := \sum_{i=0}^q (-1)^i \partial_i \quad \text{for free abelian group}$

$q$-simplex $[0,1]^n \cap \{ \sum_i t_i = 1 \}$ with facets $\Delta^q_i := \{ t_i = 0 \}$

$\Delta^q_0 := \{ t_i = 1 \}$. The singular cohomology is just the cohomology of the dual complex. Upon extending coefficients to a field $\mathbb{F}(= \mathbb{Q}, \mathbb{R}, \mathbb{C}, \text{etc.})$, we have $H^q(M; \mathbb{F}) \cong H_q(M; \mathbb{F})^\vee$. □

Corollary 5: Let $M$ be a compact Kähler manifold with compact complex submanifold $N$. Considering the latter as a topological cycle of (real) dimension $2d$, we have

$0 \neq [N] \in H_{2d}(M, \mathbb{Z})$.

Proof: If $\omega \cdot N = \int \omega$, then

$$q(d!) \text{vol}(N) = q \int_N \omega^d = \int_M \omega^d = \frac{1}{n!} \int M^d = 0.$$
Example 1: The Hopf manifolds (compact, complex, $\mathbb{C}O(n)$)

$$M_n := \mathbb{C}^n \setminus \{0\} \cong S^{2n-1} \times S^1 \quad \text{defn}.$$ 

We do **not** Kähler for $n \geq 2$, since otherwise we would have

$$0 \neq [M_1] \in H_2(M_n) = H_2(S^{2n-1} \times S^1) = S^0.$$  

Example 2: If $\Lambda \subset \mathbb{C}^n$ is a full (rank $2n$) lattice, then

$$\omega = \frac{i}{2} \sum e_i \wedge d\bar{e}_i \quad \text{forms the complex } n-\text{torus } \mathbb{C}^n/\Lambda$$

as Kähler (with $k = \sum e_i \wedge d\bar{e}_i$). Obviously also $\mathbb{C}^n$ (= affine $n$-space) is Kähler.

**KEY EXAMPLE 3: (Projective Space)**

$$\mathbb{P}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / \langle \langle \frac{2}{z} \rangle \rangle \quad \langle \frac{2}{z} \rangle \quad \text{vector for pt}.$$  

(i) $\mathbb{C}$-manifold structure: Let $\mathcal{U}_i := \{[z] \in \mathbb{P}^n | z_i \neq 0\}$

$$\varphi_i : \mathcal{U}_i \rightarrow \mathbb{C}^n \quad [z] \mapsto \left( \frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i} \right)$$

The $\varphi_i$ are holomorphic when defined, e.g.

$$\varphi_i(w_1, w_2) = \left( \frac{1}{w_i}, \frac{w_i}{w_j} \right).$$

$$\psi_{ij}(w_1, w_2) = \left( \frac{z_0}{z_i}, \frac{z_i}{z_j} \right)$$

$$\psi_{01}(w_1, w_2) = \left( \frac{z_0}{z_1}, \frac{z_1}{z_0} \right)$$
(ii) $\mathcal{E}$- submanifolds: (a) let $F(x) \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d$ \( \not\equiv 0 \) \( \mod \mathfrak{m} \), i.e., \( F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x) \).

\[ \mathcal{X} = \{ \overline{\{ x \in \mathbb{P}^n \mid F(x) = 0 \}} \} \]

is smooth at point $P$ \( \iff \) some $\frac{\partial F}{\partial x_i}(P) \neq 0$ (otherwise, $P$ is a singular point of $\mathcal{X}$)

Since $\sum_i \frac{\partial F}{\partial x_i}(P) = d \cdot F(P) = 0$, another $\frac{\partial F}{\partial x_j}(P) \neq 0$. So $f$ \( \in \mathfrak{m} \).

If $f \in \mathfrak{m}$, then the local equation $0 = f(z) = F(1; z)$ for $\mathcal{X}$ has a nontrivial gradient at $P$, so the Rank Theorem provides a (local) chart for a nbhd. of $P$. Conclude that if $\mathcal{X}$ is smooth (at all points), then $\mathcal{X}$ is a compact complex manifold.

(b) General projective variety: Given a collection of homogeneous polynomials $F_1, \ldots, F_k$ (of various degrees),

\[ \mathcal{X} = \{ z \mid F_1(z) = \cdots = F_k(z) = 0 \} \]

is smooth of codimension $c$ at $P \iff$ $\nabla F_{i_1, \ldots, i_c} \neq 0$ at $P$ for each $i_1, \ldots, i_c \in \{1, \ldots, k \}$.

\[ \mathcal{X} \cap W = \{ z \mid F_{i_1, \ldots, i_c}(z) = 0 \} \cap W \]

AND
\[ \text{rank} \left( \begin{array}{c|c} \frac{\partial F_{i_1, \ldots, i_c}}{\partial z_j} & 0 \\ \hline 0 & \frac{\partial F_{i_1, \ldots, i_c}}{\partial z_j} \end{array} \right) \bigg|_{j = 0, \ldots, n} \]

\[ c \]

* I use the notation $\mathcal{X}$ for projective algebraic varieties, which for present purposes are varieties (i.e., $V(\mathfrak{a})$) of homogeneous ideals in $\mathbb{C}[x_0, \ldots, x_n]$. Scheme theory gives a more intrinsic characterization (like what we have for manifolds).
If $X$ is smooth of the same dimension at each point, the Poincaré theorem again endows $X$ with a (compact) $C^\infty$ manifold structure.

(c) A special case which includes the hypersurface example (a) above, is the case of smooth complex intersections, i.e., when $k \geq 2$ in (b) (no local renaming of eqns.).

(iii) Fubini–Study metric (or rather, the associated $(1,1)$-form)

**Notation:**
- $\|z\|^2 = \sum |z_j|^2$
- $\rho_j(z_j) = \frac{\|z\|^2}{|z_j|^2}$, $\rho_j \in C^\infty(U_j)$.
- $\omega_j := \frac{1}{2\pi i} \bar{\partial} \partial \log(\rho_j) \in A^{1,1}(U_j)$.

On $U_{j,k}$, $\log(\rho_j) = \log \frac{z_k}{z_j} + \log \left( \frac{\bar{z}_k}{\bar{z}_j} \right)$, but

$$\bar{\partial} \partial \left( \log \frac{\bar{z}_k}{\bar{z}_j} + \log \left( \frac{\bar{z}_k}{\bar{z}_j} \right) \right) = 0 \implies \omega_j = \omega_k.$$

$$\implies$$ we have well-defined $(1,1)$ form

(E.7) $\omega = \frac{-1}{2\pi i} \bar{\partial} \partial \log(\|z\|^2) \in A^{1,1}(P^n)$

with $d\omega = (\bar{\partial} + \partial)\omega = 0$. It remains to check

**Positivity:** Let $A$ be unitary matrix,

$$P^n \ni [z] \mapsto A[z] \quad \implies \quad P^n \ni A^* \omega = \omega.$$  

The set of such transformations acts transitively on $P^n$, so it will suffice to check $\omega > 0$ at one point, say $P = [1:0:...:0] \in P^n$. In coordinates $(w_1, ..., w_n)$ we have

$\omega = \sum \frac{1}{2\pi i} \bar{\partial} \partial \log |w_j|^2$. 

Therefore, $\omega$ is positive definite.
\[
\rho_p = 1 + \sum w_i \bar{w}_i \\
\overline{\partial} \log \rho_p = \frac{\sum w_i \overline{d} w_i}{\rho_o} \\
\overline{\partial} \partial \log \rho_p = \frac{\sum \overline{d} w_i \wedge \overline{d} w_i}{\rho_o} - \frac{\left( \sum w_i \overline{d} w_i \right) \wedge \left( \sum w_i \overline{d} w_i \right)}{\rho_o^2}
\]

so

\[
\frac{\partial}{\partial \rho} = \frac{\partial}{\partial \rho} \left( \frac{\sum \overline{d} w_i \wedge \overline{d} w_i}{\rho_o} \right) = \frac{\partial}{\partial \rho} \left( \sum \overline{d} w_i \wedge \overline{d} w_i > 0 \right).
\]

Applying Corollary 3-5, we obtain

**Theorem 3**: (i) $P^n$ and all smooth projective varieties are Kähler

(ii) They have nonvanishing even-degree simplicial de Rham cohomologies up to twice their dimension.

**Remark 1**: The function $\log (\rho_p)$ in the above is called a Kähler potential. By the $\overline{\partial}\partial$-lemma (HW #2, Exercise 4), every Kähler metric may be locally described as $\overline{\partial}\partial$ of such a potential.

**Remark 2**: In fact, the Fubini–Study metric is not just Kähler but Kähler–Einstein, i.e., proportional to the Ricci curvature tensor. These are highly desirable and so difficult to find that numerical methods have come into vogue ("numerical Kähler-Ricci flow").
Here is a more general perspective on Frobenius Study.

Let $M = \mathbb{C}$-manifold

$$E \xrightarrow{\pi} M = \text{holo. vector bundle (of ex. rank r)}$$

$h = C^\infty$ Hermitian metric on $E$

Ex/using a partition of unity, prove that such an $h$ always exists.

Over $U_x \subset M$ define a basis of $C^\infty$ sections of $E$ by $\sigma^\alpha_j(p) = \Phi_X^{-1}(p, e_j)$

(Where $\Phi_X : \pi^{-1}(U_x) \to U_x \times \mathbb{C}^r$ and $e_j = (e_{1j}, \ldots, e_{nj}) \in \mathbb{C}^r$)

and $h|_{U_x}$ is determined by the $C^\infty$ functions $h_{ij}^\alpha(p) = h(\sigma_j(\Phi^{-1}_x(p), \sigma_i(\Phi^{-1}_x(p))))$.

If $r = 1$ then $E = \mathbb{L}$ is called a line bundle, and

we write $\sigma^\alpha_1 = \sigma^\alpha$, $h_{11}^\alpha = \rho_x$. We have on $U_x \beta$

$$\rho_x = h(\sigma_1, \sigma_\beta) = h(\Phi_{x\beta} \Phi^{-1}_x \sigma^\alpha_1, \Phi_{x\beta} \Phi^{-1}_x \sigma^\alpha_\beta) = |\Phi_{x\beta}|^2 h(\Phi^{-1}_\beta \sigma^\alpha_\beta, \Phi^{-1}_\beta \sigma^\alpha_\beta) = |\Phi_{x\beta}|^2 \rho_\beta,$$

where (cf. Defn C-2) the transition function $\Phi_{x\beta}$ is $\mathcal{O}(\mathbb{C}^k \times U_x \beta)$.

Thus $h_{ij}^\alpha$ are meromorphic.

It follows that $\overline{\partial} \log \rho_x = \overline{\partial} \log \rho_\beta$ on $U_x \beta$, and so

$$\omega_{\alpha} \equiv \left\{ \frac{1}{2\pi i} \overline{\partial} \partial \log(\rho_x) \right\}_{\alpha} \in \Omega^{1,1}(M)$$

defines a global real $(1,1)$-form which is also $\overline{\partial}(\partial + \overline{\partial})$-closed.

If $\tilde{h}$ is another Hermitian metric, with (say) $\tilde{h} - h$

supported over $U_x$, then $2\omega : (\overline{\partial} - \omega) = 2 \overline{\partial} \log \frac{\tilde{h}_x}{h_x} = \partial(\log \frac{\tilde{h}_x}{h_x})$

is exact. (You can't show $\omega$ exact in this way,

\[ \frac{\partial}{\partial x} \frac{\overline{\partial} x}{h_x} \]

* $\log \rho_\beta = \log \rho_x + \log \Phi_{x\beta} + \log \overline{\Phi_{x\beta}}$
because the $\{\delta(\log p_x)\}$ don't "piece together" globally.

**Definition 4:** \( c_i(L) := [\omega_{(\nu, \eta)}] \in H^{2}_{DR}(\mathbb{M}, \mathbb{R}) \), which we just checked is independent of \( \eta \), is called the **first Chern class** of the line bundle \( L \).

Example 4: (i) Define the tautological line bundle on \( \mathbb{P}^n \) by

\[
\mathcal{O}(-1) := \{ ([\omega_i], \nu) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid \nu \in \langle \omega_i \rangle \} \\
\begin{tikzcd}
\mathbb{P}^n \ar[r] & \prod_{i=1}^{n+1} \mathbb{C} \\
\pi_i \ar[u] \ar[d] & \\
\mathbb{C} \ar[u] & \\
\end{tikzcd}
\]

\( \pi^{-1}([\omega_i]) \xrightarrow{\phi} \mathbb{U}_i \times \mathbb{C} \cong \mathbb{C}^{n+1} \)

\( ([\omega_i], \nu) \quad \mapsto \quad (\phi_i([\omega_i]), \nu_i) \)

\( \Rightarrow \) \( \phi_i([\omega_i]) = \frac{\omega_i}{\nu_i} = \frac{\omega_i}{\nu_j} \)

(ii) Write \( \mathcal{O}(1) := \mathcal{O}(-1)^\vee \Leftrightarrow \phi_i = \frac{\omega_i}{\omega_j} \)

\( \mathcal{O}(a) := \mathcal{O}(1)^{\otimes a} \quad \Leftrightarrow \quad \phi_i = \frac{\omega_i^a}{\omega_j^a} \)

Given \( P \in S_{m+1}^a \) (homog. poly.), set

\( f_i := \frac{P}{\omega_i} \in \mathcal{O}(\mathbb{U}_i) \)

In \( \mathbb{U}_{ij} \), \( \mathbb{Z}^a f_i = P = \mathbb{Z}^a f_j \)
\[ f_i = \left( \frac{z_j^a}{\bar{z}_i^a} \right) f_j, \Rightarrow P \in O(1^n, O(a)). \]

Indeed, \( O(1^n, O(a)) \cong \mathbb{R}^n \).

Example 5: The canonical line bundle on an \( n \)-dimensional complex manifold \( M \) is the holomorphic bundle

\[ K_M = \Lambda^n T^{(1,0)}_M. \]

We have \( \mathcal{O}^n = \mathcal{O}(M, K_M) = \) top degree holomorphic forms.

Proposition 1: A compact complex manifold \( M \) can be a projective variety ONLY IF \( M \) admits a positive holomorphic line bundle.

\[ [i^* \omega_{FS}] = c_1(O_{\mathbb{P}^n}(1)) \quad \text{and} \quad i^* \omega_{FS} = \omega(O_{\mathbb{P}^n}(1, h_{\mathbb{P}^n})). \]
Ex/ By examining transition functions, show that
\[ K_{\mu \nu} \sim \Theta(-n-1). \]

Here is one more beautiful fact about Kähler manifolds.

**Proposition 2:** Let \( M \) be compact Kähler. Then
\[ L^q(M) \subset H^{q,0}(M, \mathbb{C}) \quad \forall q = 0, \ldots, n. \]

**Proof:** Let \( \{ \varphi_0, \ldots, \varphi_n \} \) be a local unitary coframe \((c A^0(\mathfrak{U}))\)
— these are NOT differentials of holomorphic coordinates. Say
\[ 0 \neq \eta = \sum I \varphi_I \in L^q(M). \]

Then \( \eta \overline{\eta} = \sum I \varphi_I \overline{\varphi_I} \)
\[ \omega = \frac{i}{2} \sum J \varphi_J \wedge \overline{\varphi_J} \Rightarrow -\omega^{-q} = C \sum_k \varphi_k \wedge \overline{\varphi_k} \]
\[ \int_M \eta \wedge \overline{\eta} \wedge \omega^{-q} = C \int_M |\eta_I|^2 \text{dvol}(g) \neq 0. \]

Can do this
\[ \text{by compact.} \]

Now suppose \( \eta = d\psi \). Then \( \int_M \omega^{-q} = 0 \Rightarrow \int_M \omega^{-q} = 0 \) \( \Rightarrow \) \( \omega \) Kähler \( \Rightarrow d\omega = 0 \)
\[ \int_M \eta \wedge \overline{\eta} \wedge \omega^{-q} = \int_M d(\eta \wedge \overline{\eta} \wedge \omega^{-q}) = 0, \quad \text{contra dictum.} \]

Finally, suppose \( d\eta \neq 0 \). But \( d\eta \in L^{q+1}(M) \), and then
the above argument shows that \( d\eta \) cannot be exact. \( \Box \)

\[ \star \]

\( \star \) for the Kähler metric \( h \).
So far we've had the following examples of Kähler manifolds:

- projective space \( \mathbb{P}^n \)
- smooth projective variety \( \overline{V}(K) \subseteq \mathbb{P}^n \), \( K \subseteq \mathcal{O}([z_0, \ldots, z_n]) = \mathbb{C}[z_{n+1}] \) of homogeneous prime ideal
- affine space \( \mathbb{A}^n \) (\( = \mathbb{C}^n \))
- smooth affine variety \( \overline{V}(J) \subseteq \mathbb{A}^n \), \( J \subseteq \mathcal{O}([e_1, \ldots, e_n]) \) a prime ideal
- complex tori
- Riemann surfaces: Riemannian real 2-manifold \( \leftrightarrow \) complex 1-manifold

Here are a couple more.

**Example 6:** \( E \) holomorphic vector bundle with Hermitian metric \( h \)

\[ \begin{array}{c}
\longrightarrow \\
E \mapsto \text{hol. vector bundle with Hermitian metric } h \\
\medskip
\end{array} \]

\( \Theta(E)(-1) \) the twisted tensor bundle

\[ \begin{array}{c}
\downarrow \\
\Theta(E)(-1) \text{ the twisted tensor bundle} \\
\medskip
\end{array} \]

\( \mathcal{P}(E) \) the fiberwise projection of \( E \)

\[ \begin{array}{c}
\left( \mathcal{P}(E) \right)_{M} \\
\medskip \\
\end{array} \]

Restoring \( h \) to \( E \) via \( \Theta(E)(-1) \) produces a metric.

Let \( \omega_E := - (\text{its Chern form}) \), which pulls back to \( \omega_F \) (hence is > 0) on each fiber; if \( \omega_F \) is still negative "in the horizontal direction," but this negativity is bounded below since \( M \) is compact. So \( \omega_E + \lambda \omega_M > 0 \) for \( \lambda > 0 \) suitably large.

* in both cases necessarily finitely generated by the Hilbert basis theorem.
Example 7: \( \frac{N}{M} := \text{Im}^{-1} \) in \( N \subset M \) (complex manifold) "normal bundle" (holomorphic)

There exists a construction called the blow-up of \( M \) along \( N \):

\[
\begin{align*}
B_N(M) & \text{ with } \begin{cases} 
\beta^{-1}(N\backslash M) = M \\
\beta^{-1}(N) = \mathbb{P}(N_N/M) \text{ - dimension } = \text{dim}_{\mathbb{C}}(M) - 1.
\end{cases}
\end{align*}
\]

Locally (on \( U \subset M \)), the idea is simple (see Voisin for the full details):

\[
\begin{align*}
N \cap U &= \{ u \in U \mid f_i(u) = 0 \quad (i=1,...,k) \} \subset U \\
\text{then } \beta^{-1}(U) &= \left\{ (u, [z]) \in \mathbb{P} \left[ \mathbb{C}^k \right] \mid z_i f_i(u) = z_j f_j(u) \quad (\forall i, j = 1,...,k) \right\}.
\end{align*}
\]

To get anything, you need \( k \geq 2 \). The prototypical example is:

\[
\begin{align*}
B_{(0,0)}(\mathbb{C}^2) &= \left\{ ((u_0, u_1), [z]) \in \mathbb{C} \times \mathbb{P} \mid z_0 u_0 = z_1 u_1 \right\} \\
&\text{where } (0,0) \text{ has been "blown up" into a } \mathbb{P}^1,
\end{align*}
\]

suggesting lines near the origin by slope —

useful if you want to make a holomorphic function like \( u/\sqrt{u} \) well-defined

Terminology: going \( M \to B_N(M) \) is "blowing up \( N \)"

\[
\begin{align*}
giving B_N(M) \to M \text{ is "blowing down } \beta^{-1}(N)".
\end{align*}
\]

Blowing up is always possible and preserves the Kähler property;

blowing down (a given submanifold to something of lower dimension) is

only possible under special circumstances and need not preserve
the Kähler property even when it is possible.

\( \square \)

\( \text{At } \) not starting from the assumption that it is \( \beta^{-1}(N) \).
We now turn to a discussion of the 3 conditions which are equivalent to the Kähler condition $\omega = 0$. To this end we first pass back to the general situation of a smooth vector bundle $E$ over a smooth manifold $M$.

**Definition 6:** (i) A **connection** on $E$ is an $\mathbb{R}$-linear map

$$\nabla : C^\infty(M,E) \to C^\infty(M, T^*_M \otimes E) =: \Lambda^1(M,E)$$

s.t. $\nabla(f \sigma) = df \otimes \sigma + f \nabla \sigma \quad (\forall f \in C^\infty(M))$.

This gives rise to **directional derivatives** (in direction $\eta \in C^\infty(M, T^*_M)$)

$$\nabla_\eta : C^\infty(M,E) \to C^\infty(M,E)$$

$$\sigma \mapsto (\nabla_\eta \sigma)(\eta).$$

(ii) If $\sigma_1, \ldots, \sigma_r$ is a basis of sections (over $U \subseteq M$), the matrix of connection 1-forms is defined by

$$\nabla \sigma_i = \sum_k \Theta^k_i \otimes \sigma_k,$$

where $\Theta^k_i \in \Lambda^1(U)$.

(iii) A section $\sigma$ is **flat** if $\nabla \sigma = 0$.

**Example 8:** The Levi-Civita connection on the tangent bundle of a Riemannian manifold $(M, g)$.

Let $E = T^*_M$, with local coordinates

$\{x^i\}$ on $U$ (and hence local sections $\{\xi_i = \partial_i\}$ of $T^*_M$).

Define the **Christoffel symbols** by

$$\Theta^k_i = \frac{\partial \Gamma^k_{ij}}{\partial x^i} dx^j$$

and the **torsion** of the connection by

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}.$$
Definition 7: The Levi-Civita connection on $T_m$ is the unique connection with $\nabla = 0 \Rightarrow \nabla^\tau \xi \cdot \nabla^\tau \xi = \nabla^\tau \nabla^\tau \xi \cdot \nabla^\tau \nabla^\tau \xi$ which is compatible with $g$:

$$d (g(\xi, \eta)) = g(\nabla^\tau \xi, \eta) + g(\xi, \nabla^\tau \eta).$$

To see that it exists and is unique, derive the formula: let $\{x^k\}$, $\{y^j\}$,

$$d (g(\xi, \eta)) = g\left(\frac{\partial y^j}{\partial x^k} \xi, \eta\right) + g\left(\xi, \frac{\partial y^j}{\partial x^k} \eta\right)$$

which is visibly symmetric in i and j.

Multiplying both sides by the inverse matrix $g^{ij}$ and summing over $k$, we get

$$(E.9) \quad \sum_{k} g_{ij} \frac{\partial y^j}{\partial x^k} = \sum_{k} g^{ik} g^{jk}$$

which is visibly symmetric in i and j.

Example 9: The Chern connection on a holomorphic Hermitian vector bundle $(E,h)$ over a $C^\infty$-manifold $M$. [cf. pp. 22-23 for defn. of $\delta_F$]

We have to refine defn. 6 a bit: start with a $C^\infty$-linear

$$\nabla: C^\infty (M, E) \to C^\infty (M, (T^*_M \otimes \mathcal{C}) \otimes E) = \Lambda^{0,0} (M, E) \otimes \Lambda^{0,0} (M, E)$$

Taking $\sigma_1, ..., \sigma_r$ to be holomorphic sections, write
\[ \nabla^{(1,0)} \sigma_i = \sum \Theta^k_i \Theta^k \]
\[ \nabla^{(0,1)} \sigma_i = \sum \tau^k_i \Theta^k \]

where \( \Theta^k_i \in A^{0,0}(\mathcal{M}) \) and \( \tau^k_i \in A^{0,1}(\mathcal{M}) \).

And notice \( \nabla^{(0,1)} = \overline{\nabla} \quad (\Leftrightarrow \tau = 0) \).

**Definition 8:** The Chern connection on \((E, h)\) is the unique (complex) connection with \( \overline{\nabla} = \overline{\nabla} \) which is compatible with \( h \):

\[ d(h(X, \psi)) = h(\nabla X, \psi) + h(X, \nabla \psi) \]

Again, we derive the formula by taking \( X = \sigma_i \), \( \psi = \sigma_j \) and noticing that

\[ d(h(X, \psi)) = h(\nabla^{1,0} X, \psi) + h(X, \nabla^{0,1} \psi) \]

\[ = \overline{\nabla} \psi - \overline{\nabla} (\sigma) = 0 \]

Assume since \( \overline{\nabla} \) is a Hermitian.

So letting \( \partial / \partial z_j \) operate on both sides and writing \( \Theta^k_i = \sum \mu^k_{ij} \partial \sigma_j \),

\[ \frac{\partial \mu^k_{ij}}{\partial z_j} = h(\nabla_{\partial / \partial z_j} \sigma_i, \sigma_j) = \sum h_{kij} \mu^k_{il} \Rightarrow \]

\[ \mu^k_{il} = \sum h_{kij} \left( \frac{\partial \sigma_j}{\partial z_l} \right) \]

\[ \text{(E.11)} \]

Now let \( E = T_M \) (holomorphic tangent bundle), with Hermitian metric \( h \), \( d g = \text{Re}(h) \), \( \omega := -\text{Im}(h) \). We can consider both the Levi-Civita \( \nabla \) for \((TM, g)\) and Chern \( \nabla \) for \((TM, \overline{\nabla})\),

where in the latter case the \( \sigma_i = \overline{\partial} z_i \) come from the choice of local coordinates.
Theorem 4: TFAE:

(i) $\omega = 0$ (h is Kähler)

(ii) h osculates to order 2 the Euclidean metric everywhere

(iii) $\nabla_{bc} = \nabla_{\text{Chern}}$

(iv) $J$ commutes with $\nabla_{bc}$.

Proof: (i) $\Rightarrow$ (ii). At a point $p$, we can write holomorphic coordinates such as $w = \frac{i}{2} \sum \delta_{ij} \overline{z}_i \overline{z}_j$ at that point. More precisely,

$$w = \frac{i}{2} \sum_{ijkl} (\delta_{ij} a_{jkl} \overline{z}_i \overline{z}_j + \overline{a}_{ijk} \overline{z}_k + O(dz^2)) \overline{dz}_i \overline{dz}_j$$

henceforth called $[z]$.

Now

$$w = \overline{w} = \frac{1}{2} \sum_{ijkl} (\delta_{ij} a_{jkl} \overline{z}_i \overline{z}_j + \overline{a}_{ijk} \overline{z}_k) \overline{dz}_j \overline{dz}_i + [z]$$

Therefore

(E.1)

$$(a_{jkl} = a_{jkl})$$

Moreover,

$$0 = dw = \frac{i}{2} \sum (a_{ijk} dz_i \wedge dz_j \wedge \overline{dz}_k + a_{jik} \overline{dz}_j \wedge dz_k \wedge \overline{dz}_i) + [1]$$

(E.2)

$$(a_{jkl} = a_{jkl})$$

Defining coordinates $(\tilde{w}_m)$ by

$$\tilde{w}_k = w_k - \frac{1}{2} \sum_{lm} a_{klm} w_l w_m$$

we have by (E.2) $dz_k = dw_k - \sum a_{klm} w_l dw_m$. So
\[
\frac{2}{i} \omega = \frac{\xi}{i} (dw_i - \frac{\xi}{m_i} w_i dw_m) \wedge (d\bar{w}_i - \frac{\xi}{\rho_i} \bar{w}_i d\bar{w}_m) \\
+ \sum_{i,j,k} (a_{ijk} w_i + a_{ijk} \bar{w}_i) dw_i d\bar{w}_j + [2] \\
= \sum_{i,j} (\xi_{i,j} + \sum_{k} (a_{ijk} w_i + a_{ijk} \bar{w}_i) - a_{ijk} \bar{w}_i - a_{ijk} w_i) dw_i d\bar{w}_j \\
+ [2] \\
= \sum_{i,j} dw_i d\bar{w}_j + [2].
\]

\((ii) \Rightarrow (iii)\). In the conditions \(\{w_j\}\) above, \(h_{ij}(p) = g_{ij}(p) = \delta_{ij}\) and the 1st partials of the \(h_{ij}\) resp. \(g_{ij}\) at \(p\) are all zero. Hence, at \(p\), \(\rho_i(p) = 0 = \rho_i(p) \Rightarrow (\nabla(\delta_{ij}))(p) = 0\) for both \(\nabla_{lc}\) and \(\nabla\text{chem}\). Since the last statement does not depend on the choice of \(\lambda\)-coordinates (by uniqueness of the connections \(\nabla_{lc}\) and \(\nabla\text{chem}\)), it is true at all points.

\[(iii) \Rightarrow (iv)\). \(\nabla_{lc} = \nabla\text{chem}\) is \(\mathbb{C}\)-linear, hence commutes with \(J\).

\((iv) \Rightarrow (c)\). Let \(\nabla = \nabla_{lc}\). Then for \(\xi, \chi\) vector fields:
\[
d(g(\xi, \chi)) = g(\nabla(\xi), \chi) + g(\xi, \nabla(\chi))
\]
\[
\downarrow (\text{apply another vector field } \eta) \\
d(\omega(\xi, \chi)) = \omega(\nabla(\xi), \chi) + \omega(\xi, \nabla(\chi))
\]

\[(\xi, \xi)\]
By the HW Exercise on p. 6 (3I.B),
\[
d\omega(\eta, \xi, \chi) = \left\{ \begin{array}{l}
\gamma(\omega(\xi, \chi)) + 3(\omega(\eta, \xi)) + \chi(\omega(\eta, \xi)) \\
\omega([\eta, \xi], \chi) - \omega([\xi, \eta], \chi) - \omega([\eta, \xi], \chi)
\end{array} \right.
\]
\[
= 0 \quad \text{after applying \ (E.14) and its two cyclic permutations.}
\]

Final remarks on how the Kähler property sits in the pantheon of complex manifolds.

Let \( M = C^n \)-manifold.

**Definition:** \( M \) is **Stein** \( \iff \) holomorphic embedding
\[
\Psi : M \to C^n \quad \text{st.} \quad \Psi(M) \quad \text{is closed in} \quad C^n.
\]

**Remarks:**
(i) By Corollary 4 (i), \( M \) is Kähler
(ii) Maximum modulus principle \( \Rightarrow M \) noncompact
(iii) Smooth affine varieties are Stein.
Now restrict to $M$ compact $C^*$-manifold.

**Definition 10:**
(i) $M$ is Moishezon $\iff$ $\text{deg} \, \mu_c(M) = \dim_{\mathbb{C}}(M)$

(ii) $M$ is Fujiki $\iff$ blow-down of $\mathbb{K}ahler$.

We then have the nice picture

![Diagram showing relationships between compact $C^*$-manifolds, Moishezon, Fujiki, and $\mathbb{K}ahler$ varieties.]

We also note the fundamental results:

**Chow's Theorem:** $M \subset \mathbb{P}^N$ complex submanifold $\iff$ smooth projective variety

("global analytic $\iff$ global algebraic")

**GAGA**

** Kodaira Embedding Theorem:** $M$ admits positive holomorphic line bundle $\iff$ $M$ is projective.
Examples: (a) 5 3 all compact Riemann surfaces
(b) 1 \( \bar{2} \)  Hopf manifolds
(c) 3 5 some Calabi-Yau 3-folds studied by physicists
(d) 2 \( \bar{5} \) "most" complex tori (the ones w/o a positive line bundle).

These complex tori in 5 (admitting a projective embedding) are called Abelian varieties.

We will have more to say about (a) \& (d) later.