We now study some applications of sheaf cohomology and the de Rham theorem in a classical setting, partly to have some concrete examples of Hodge decompositions of bilinear relations before we meet them in greater generality.

As defined above, a compact Riemann surface (CRS) \( M \) is just a compact complex 1-manifold. \( M \) is Kähler, of course, since \( dw = 0 \) automatically (no 3-forms on \( M \)). We will see below that it is in fact projective.

The isomorphism class of \( M \) as a \( C^\infty \) manifold is given by the genus \( g \) of \( M \):

\[
g = 0 \quad \quad \quad \quad g = 1 \quad \quad \quad \quad g = 2
\]

and this determines its homology \( H_1(M, \mathbb{Z}) \cong \mathbb{Z}\langle x_2, \ldots, x_g \rangle \):

Note the (perfect) intersection pairing

\[
(\xi, \eta) \quad H_1(M, \mathbb{Z}) \times H_1(M, \mathbb{Z}) \to \mathbb{Z}
\]

with matrix

\[
J = \begin{pmatrix}
0 & \text{I}_g \\
\text{I}_g & 0
\end{pmatrix}
\]

Aside: Note that the subgroup of \( GL_2(\mathbb{Z}) \) defined (wrt.

basis \( \{x_i\} \) if you will) by

\[
\text{M}_g \mathbb{Z} = J
\]

is called the symplectic group

\( \text{Sp}_g(\mathbb{Z}) \).
As for the cohomology, we have the cup (wedge) product
\[ (6.2) \quad H^1_{dR}(M, \mathbb{C}) \times H^1_{dR}(M, \mathbb{C}) \to H^2_{dR}(M, \mathbb{C}) \xrightarrow{\cup} \mathbb{C} \]
as well as the (perfect) pairing induced by integration
\[ (6.3) \quad H^1_{dR}(M, \mathbb{C}) \times H_1(M, \mathbb{C}) \to \mathbb{C} . \]
We get a composite isomorphism
\[ H_1(M, \mathbb{C}) \xrightarrow{(6.1)} H^1(M, \mathbb{C}) \xrightarrow{(6.3)} H^1_{dR}(M, \mathbb{C}) \]
which begs the question: are the pairings all compatible, i.e., does
\[ \int \quad \cdots \quad \int \quad \text{commute?} \]
In fact, the only issues here is the "top triangle": that is, do we have
\[ \int_M \eta_Y \wedge \eta_{Y'} = [\gamma] \wedge [\gamma']? \]
There is a simple construction for \( \eta_Y \in A^1(M) \)-closed which makes this obvious: take a \( C^\infty \) embedding of \( |Y| \times \Delta \) in a small tubular neighborhood of \( |Y| \), which gives a diagram
\[ \begin{array}{ccc}
|Y| \times \Delta & \to & M \\
\downarrow \pi & & \\
\Delta & & \\
\end{array} \quad (\Delta = \text{unit disk}) \]
Taking a $C^\infty$ bump 1-form $\eta$ on $\Delta$ with $\int_\Delta \eta = 1$, we set $\eta_y := \iota_{\pi^* \eta}$. By Fubini, one sees that at intersections

$$\int \eta_x \wedge \eta_y = \pm \int \pi^* \eta \wedge \pi^* \eta = \pm \left( \int_\Delta (\pi^* \eta)^2 \right) \quad \text{according to (6.2) is positive),}$$

which $\Rightarrow (6.4)$ commutes (and (6.2) is positive).

So periods of $C^\infty$ 1-forms are whatever we want them to be. Holomorphic forms are more interesting. To see this we will require the following "preliminary" result:

Poincaré–Hopf Theorem: (i) Given $\omega \in C^\infty(M, TM)$, $\sum_{p \in M} \text{ind}_p \omega = \chi_M$.

(ii) Given $\omega \in \Omega^1(M)$,

\[\sum_{p \in \Sigma} \text{ind}_p \omega - \text{# of zeroes} = 2g - 2.\]

Proof: (i) "triangulate" $M$, and draw the following vector field on each triangle.

Which clearly gives together to give a global vector field on $M$, with indicies $\pm 1$.

The marked points on the edges and $+1$ at the marked points on the faces + vertices. Thus,

$$\sum \text{ind}_p \omega = \# F - \# E + \# V = \chi_M = 2 - 2g.$$
It's fairly easy to see that this invariant is continuous under (co) derivation, hence that (6.5) holds for any vector field \( \tilde{v} \). It still holds if we allow \( \tilde{v} \) to have singularities at a finite number of points \( \{p_1, \ldots, p_n\} \) (i.e. \( \tilde{v} \in \mathcal{C}^0(M|\cup_{i=1}^n \{p_i\}, T_M) \)) provided one adds in the indices of \( \tilde{v} \) at the \( p_i \) into the sum.

(ii) (6.5) even holds if \( \tilde{v} \) is replaced by a smooth \(-\)form \( \eta \in \Gamma(M|\cup_{i=1}^n \{p_i\}) \), by using a metric \( g \) to identify \( TM \cong \mathbb{T}M \).

The corresponding notion of index, if \( \eta = \int f \, d\gamma + G \, dy \), is

\[
\text{Ind}_p \eta := \frac{1}{2\pi} \oint \frac{d\alpha}{\cos(\alpha - \gamma)}
\]

and once again the sum in (6.5) must be over all \( \mathcal{O}'s \) of \( \eta \).

If we \( K(M) \) has local form \( \omega = f \, dx + g \, dy \) (\( f, g \) are valued in \( \mathbb{C} \)), then \( \eta := \text{Re}(\omega) = \text{Re}(f) \, dx + \text{Re}(g) \, dy \). Let \( p = 0 \) be a pole of \( \omega \), and put \( \nu_p := \text{ord}_p(\omega) \). In a local holomorphic chart \( \mathcal{O} \) at \( p \),

\[
\omega \equiv r\, (\cos \nu \theta + i \sin \nu \theta) \,(dx + idy)
\]

\[
= r^\nu (\cos \nu \theta - i \sin \nu \theta)) \, dx + r^\nu (\sin \nu \theta + i \cos \nu \theta) \, dy.
\]

For the real part, then,

\[
\eta \approx \cos(-\nu \theta) \, dx + \sin(-\nu \theta) \, dy
\]

and so by (6.6)

\[
\text{Ind}_p \eta = \frac{1}{2\pi} \oint \! d[-\nu \theta] = -\nu_p
\]

\[
\Rightarrow \sum_{p \in M} \nu_p = 2g - 2.
\]

The compactness result for a meromorphic function \( f \in \mathcal{M}(M) \backslash \{0\} \) is

\[
\left( \# \text{zeros of } f \right) - \left( \# \text{poles of } f \right) = \sum_{p \in M} \text{Res}_p \frac{df}{f} = \frac{1}{2\pi i} \oint_{\mathcal{M} \cup \cup_{i=1}^n \{p_i\}} \frac{df}{f} = \frac{1}{2\pi i} \oint_{\mathcal{M} \cup \cup_{i=1}^n \{p_i\}} d(f/\bar{f}) = 0
\]

up to multiplication by a local holomorphic \( \tilde{v} \), which will not affect index.
We can rephrase these statements in terms of the group of (well) \textit{divisors}:

\[
\text{DIV}(M) := \left\{ \sum_{p \in M} d_p [p] \mid d_p \in \mathbb{Z}, \text{ finitely many } p \right\}
\]

\[
\downarrow \text{deg (= degree homomorphism)}
\]

\[
\mathbb{Z} \rightarrow \sum d_p =: d
\]

Write \( \text{Div}(M)^0 := \ker(\text{deg}) \)

- \( D \) is effective \( \Leftrightarrow d_p \geq 0 \) \((\forall p) \Rightarrow "D \geq 0"
- \( D \geq E \Leftrightarrow D - E \geq 0
- A\text{ given } D\text{ may be written } D_+ - D_-, \text{ with } D_+, D_- \geq 0.

\text{Example 1: (i) } f : \sum_{p \in M} n_p(f)[p] \text{ induces a group homomorphism }

(\cdot) : M(M)^* \rightarrow \text{DIV}(M)^*

(ii) Given \( \omega \in K^1(M)^* \), writing \( \omega = f dx \) in local coordinates one defines \( \nu_p(\omega) := \nu_p(f) \), this yields

\( \omega := \sum_{p \in M} \nu_p(\omega)[p] \), and Poincaré--Hopf \( \Rightarrow \)

\[ \text{deg}(\omega) = 2g - 2. \]

We now have the crucial

\text{Definition 1: } \mathcal{O}(D) := \text{sheaf of meromorphic functions } f \text{ satisfying } (f) + D \geq 0 \text{ (or } f \equiv 0 \text{ locally)}
\[ \mathcal{O}^1(D) := \text{sheaf of zero- \textit{locally} } 1 \text{-forms satisfying \textit{(locally)}} \]
\[ (\omega + D) \geq 0 \quad \text{(or } \omega = 0) \]
\[ L(D) := \mathcal{O}^1(D) = H^0(\mathcal{O}(D)) \quad \text{and } L(D) := H^1(\mathcal{O}(D)) \]

(the "m" is understood)
\[ l(D) := \dim L(D) \quad \text{and } i(D) := \dim \mathcal{O}(D). \]

For \( D \geq 0 \), one has short-exact sequences

\[ 0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(D) \rightarrow \bigoplus \mathbb{C}^d \rightarrow 0 \quad \text{(reduces principal part)} \]

\[ 0 \rightarrow \mathcal{O}^1 \rightarrow \mathcal{O}^1(D) \rightarrow \bigoplus \mathbb{C}^d \rightarrow 0 \]

with associated long-exact sequences

\[ 0 \rightarrow \mathcal{O}(\mathbb{M}) \rightarrow L(D) \rightarrow \mathbb{C}^{d(\deg D)} \rightarrow H^1(\mathcal{O}) \rightarrow L(D) \rightarrow 0 \]

\[ 0 \rightarrow \mathcal{O}^1(\mathbb{M}) \rightarrow \mathcal{O}^1(D)(\mathbb{M}) \rightarrow \mathbb{C}^d \rightarrow H^1(\mathcal{O}^1) \rightarrow H^1(\mathcal{O}^1(D)) \rightarrow 0 \]

in which the alternating sum of dimension must be zero (exercise).

For \( D > 0 \) we also have

\[ 0 \rightarrow \mathcal{O}(-D) \rightarrow \mathcal{O} \rightarrow \bigoplus \mathbb{C}^d \rightarrow 0 \]

\[ 0 \rightarrow \mathcal{O}(\mathbb{M}) \rightarrow \mathbb{C}^d \rightarrow \mathcal{O}(-D) \rightarrow H^1(\mathcal{O}) \rightarrow 0 \]

(hold for vanishing somewhere on \( D \))

and so for \( d > 0 \)
\[(G.12)\] \[
\begin{align*}
\ell(D) &\geq d + O(1) \quad (\text{in the sense of asymptotics:}) \\
\dim R^1(D)(M) &\geq d + O(1) \\
i(-D) &= d + O(1)
\end{align*}
\]
This immediately gives

Proposition 1: Nonconstant meromorphic functions and nonzero meromorphic forms exist.

Let \( \omega \in K^1(M)^* \) then, and set

\[ K := (\omega) \in \text{Div}(M), \]
this canonical divisor is well-defined modulo \((\mathcal{M}(M)^*)\).

Corollary 1: \( \Theta(K) \leq \Lambda^1 \).

\[ \text{Proof: } f \mapsto f\omega, \quad \text{check: } (f\omega) = (f) + (\omega) = (f) + K \geq 0, \]
\( \forall \omega \rightarrow ? \)

Now let \( D = D_+ - D_- \), \( \sum_{D_+} 0 \)

\( D_- > 0 \Rightarrow \)

\( 0 \to \Omega(-D_-) \to \Omega(D) \to \bigoplus \mathfrak{g} \mathfrak{c} \to 0 \Rightarrow \)

\[(G.13)\]

\[ 0 \to \ell(D) \to \mathfrak{g}^* \to \lambda(-D_-) \to \ell(D) \to 0 \]

Now \((G.9) \Rightarrow \)

\[ \ell(D) - i(D) = d + 1 - g_a \quad \text{for } D \geq 0, \]
where \( g_a := \dim H^1(\mathcal{O}) \). For the remaining cases of \( D \),
\[(G.13) \quad \tilde{\rho}(D) - i(D) = \tilde{d}^+ - i(-D) = \tilde{d}^+ - (\tilde{d}^- + g_\omega - 1) = \tilde{d}^+ - g_\omega.\]

**Riemann-Roch Theorem:** For all \(D \in \text{Div}(M),\)

\[\tilde{\rho}(D) - i(D) = \tilde{d}^+ - g_\omega.\]

**Remark 1:** While (6.12) was proved for \(D \geq 0,\) it's clear now that it holds in general. (For \(i(-D),\) this means showing \(\tilde{\rho}(-D) = 0\) for \(D\) sufficiently large; but already for \(D > 0\) \(\tilde{\rho}(-D)\) is zero for a deg \((C_f)\) = 0 always.)

To understand \(\tilde{\rho}(D)\) and \(g_\omega,\) we have to do some heavy lifting. Multiplication induces a map of sheaves

\[\mathcal{O}(-D) \otimes \mathcal{L}(D) \to \mathcal{O}^1,\]

hence a pairing

\[H^1(\mathcal{O}(-D)) \otimes H^0(\mathcal{L}(D)) \to H^0(\mathcal{O}^1) \cong \int \frac{\mathcal{L}}{\mathcal{O}^1} \to H^0_\omega(M, \mathcal{C}).\]

One can think of this as a map

\[\Theta_D : \tilde{\rho}(D)(M) \to (H^1(\mathcal{O}(-D))^\vee,\]

\[\omega \mapsto \{ \mathfrak{g} \mapsto e(\mathfrak{g} \omega) \} \]

**Theorem 1 (Serre duality, for curves):** \(\Theta_D\) is an isomorphism \((\forall D).\)
Proof: For $E \geq D$ we have
\[
0 \rightarrow \Omega(-E) \rightarrow \Omega(-D) \rightarrow \Theta \oplus \mathcal{E}^{e-d} \rightarrow 0
\]
\[
\Rightarrow \quad H'(\Omega(-E)) \rightarrow H'(\Omega(-D)) \rightarrow 0
\]
\[
\Rightarrow \quad H'(\Omega(-D)) \rightarrow H'(\Omega(-E))
\]
Set $V = \lim_{\to} H'(\Omega(-D))$.

Claim 1: $\Theta : \mathcal{H}^2(M) \rightarrow V$ is injective.

Pf: let $\omega \in \mathcal{H}^2(D(M))$
\[
k = -(1+\chi(\omega)) \quad (\leq d_p-1)
\]
Introduce a class $\delta \in H'(\Omega(-D))$ by taking
\[
\delta_{12} = z^k \in \Omega(-D)(U_{12}) \quad \text{and} \quad f_{91}
\]
is not a good open cover, this won't affect anything.

[Note: for $z^k$ to extend to $\Omega(-D)(U_1)$, translating $\delta$, we'd need $k \geq d_p$]

Let $\{ g_2 \in C^0(U_2) \}$ extend $\delta$; then $\{ \overline{\delta g} \}$ involves $\delta$ as an element of $H'(\mathcal{C}^0(M))$, and $\overline{\delta g}$ gives the differentiation of $\delta$ in $A^{0,1}(-D)$.

and
\[
\mathcal{E}(\delta g) = \int_M \omega \wedge \overline{\delta g} = \int_{MU_1} \omega \wedge \overline{\delta g_2}
\]
\[
= \int_{MU_1} d(g_2 \omega) = \int_{2U_1} 2k_z \omega = 2\pi i \operatorname{Res}_p(z^k \omega) \neq 0 \quad \text{by choice of } k.
\]
So we obtain a nonzero element of $V$. //

* in particular, recall that $H'(\Omega(-D))$ is defined as $\lim_{\to} H'(\Omega(-D))$
so one can use any $U$ to define a class. By $\delta$ we mean its image in the limit.
Claim 2 \( w \in \mathcal{K}(M) \) maps into \( H^1(\mathcal{O}(-D))^\vee \) \( \Rightarrow \) \( w \in \mathcal{N}(D)(M) \).

**PF:** If \( \Theta(w) \in \mathcal{N}(D)(M)^\vee \), then it has to vanish on all coboundaries for this group; if also \( \kappa = \nu_p(w) < -d_p \), then \( \kappa \geq d_p \Rightarrow \)
\[ \exists \gamma \in \text{above}, \text{but } (\nu_p(w) \neq 0 \text{ or } \gamma) \quad \Rightarrow \quad \nu_p(w) \geq -d_p \quad (\forall \gamma) \quad // \\
\]

It remains to prove \( \Theta \) surjective. First note

- \( 1 \)-dim \( \Gamma \) \( M(M) \) - vector space
- \( V = M(M) \) - vector space (given \( v \in V \), \( f \cdot v)(x) := v(f(x)) \)
- \( \Theta \) is \( M(M) \) - linear \((\Theta(fw))(x) := e(\omega f(x)) = \Theta(\omega)(f(x)) = (f \cdot \Theta(\omega))(x) \)

Now for \( \phi \in \mathcal{N}(\mathcal{O}(-D))^\vee \subset V \), it will suffice to prove the

Claim 3 \( \phi = \Theta(\tilde{\omega}) \) for some \( \tilde{\omega} \in \mathcal{K}(M) \).

**PF:** by (6.12) + Remark 1, for \( n > 0 \)
\[ \dim \mathcal{N}(\mathcal{O}(-D - n[p]))^\vee = n + \Theta(1) \]
Moreover, \( \mathcal{N}(\mathcal{O}(-D - n[p]))^\vee \) contains
- \( \Theta(\mathcal{N}(D + n[p])(M)) \) \( \sim \) \( \dim \geq n + \Theta(1) \) by (6.12) & Claim 1
- \( \Theta(n[p])(M) \), \( \phi \) \( \sim \) \( \dim \geq n + \Theta(1) \) by (6.12)

So for \( n \) sufficiently large, these spaces intersect nontrivially:
\[ \mathcal{P} \begin{cases} f \in \Theta(n[p])(M) \\ w \in \mathcal{N}(D + n[p])(M) \text{ s.t. } f \cdot \psi = \Theta(w) \end{cases} \]
\[ \Rightarrow \quad \psi = \Theta(\omega) = \Theta(fw), \quad \text{dim.} \quad // \\
\]

By Claim 2, \( \tilde{\omega} \) will automatically lie in \( \mathcal{N}(D)(M) \). \( \square \)
Here come the corollaries!

**Corollary 2:**

(i) \( i(D) = \dim \mathcal{R}(-D)(M) \)

(ii) \( H^0(O(D)) \cong H^1(N^1(-D)) \)

(iii) \( e : H^1(N^1) \to \mathcal{C} \) is an isomorphism

**Pf.:**

(i) is immediate from Thm. 1

(ii) \( H^0(N^1(D-K)) \cong H^0(O(K-D)) \)

(iii) is the special case \( D = 0 \) of (ii).

Now we have \( H^0(N^1)^\vee \cong H^1(O) \Rightarrow \dim \mathcal{R}^1(M) = g_a \). Moreover,

\[
0 \to C \to \mathcal{O} \to \mathcal{R}^1 \to 0
\]

Gives

\[
0 \to C \to \mathcal{O} \to \mathcal{R}^1(M) \to H^1(C) \to H^1(O) \to H^2(N^1)^\vee \cong H^2(C) \to 0
\]

\[
\Rightarrow 2g_a = 2g \Rightarrow g_a = g.
\]

**Corollary 3:**

(i) \( \dim \mathcal{R}^1(M) = g \)

(ii) \( [K-R^7] \mathcal{L}(D) - i(D) = d - g + 1 \)

(iii) \( H^2(M, C) \cong H^1(N^1) \)

**Remark 2:**

Given a finite set of principal parts \( \{ \frac{a^{(c_i)}}{z_i^{(c_i)}} + \ldots + \frac{a^{(c_i)}}{z_i^{(c_i)}} \} \) on \( M \), can we solve the Mittag-Leffler problem for \( K^1(M) \)?

Well, it's clear from the Residue theorem that a necessary condition is

\[
\sum_{i=1}^{k} a^{(c_i)} = 0.
\]
Let $\tilde{D} = \sum d_i [p_i] (\geq 0)$; then $\lambda(-D) = 0$, so by R-R
\[ \dim H^0(\tilde{D}) = g + \sum d_i - 1, \] and the dimension of the
subspace of principal parts spanned by these forms is
\[ \dim H^0(\tilde{D}) - \dim H^0(\tilde{L}) = \sum d_i - 1. \]
That means (ii) is also a sufficient condition.

Now we have a map
\[
(6.14) \quad \Xi^1(M) \oplus \Xi^1(N) \to H^1_{dR}(M/G),
\]
induced by
\[
(\omega, \bar{\psi}) \mapsto [\omega + \bar{\psi}],
\]
as well as a (compatibility) map
\[
\Xi^1(M) \to H^0(\tilde{M}),
\]
induced by
\[
(\omega, \bar{\psi}) \mapsto \omega + \bar{\psi}.
\]

**Corollary 4:** (i) [Hodge decomposition]
\[
H^1_{dR}(M/G) \cong \Xi^1(M) \oplus \Xi^1(N).
\]

(ii) \[ \Xi^1(M) \cong H^0(\tilde{M}). \]

**Proof:** (i) We need to check (6.14) injective (it then follows from
equality of dims.). Suppose $\omega + \bar{\psi} = df$, $f \in C^\infty(M)$. Then
\[
\int_{\partial M} \omega = \int_{\partial M} df = \int_{\partial M} \partial f = 0 \Rightarrow f \in \Omega^0(M),
\]
\[
\Rightarrow \omega = 0.
\]

(ii) We have a diagram (exact rows)
\[
0 \to \Xi^1(M) \to H^1(\mathcal{O}) \to H^1(\mathcal{O}) \to 0
\]
\[
\downarrow \quad \text{incl.}
\]
\[
0 \to \Xi^1(M) \to \Xi^1(M) \to \Xi^1(M) \to 0.
\]

\[
\Rightarrow \text{last arrow is an \epsilon.}
\]
Conclusion 5: (a) \( d > 2g - 2 \Rightarrow \chi(D) = 0 \)
(b) \( d < 0 \Rightarrow \chi(D) = 0 \)

Let \( D = k[p] - \sum_{i=1}^{l} [p_i] \), \( k-l > 2g-2 \). Then \( R - R + \text{Cor. 5} \Rightarrow \)

\[(6.15) \quad \chi(D) = k-l-g+1. \]

Conclusion 6: \( \exists \) holo. embedding \( g : M \hookrightarrow \mathbb{P}^{g+1} \).

Proof: Fix \( p \in M \). Then for arbitrary \( k \in \mathbb{Z} \), \( r \in M \),

\[(6.16) \quad \chi((2g+1)[p] - [q] - [r]) \cong \chi((2g+1)[p] - [q]) \cong \chi((2g+1)[p]) \]

\[
\begin{array}{c|c|c|c}
\text{degree} & \frac{2g-k}{2g+1} & \frac{2g-1}{2g} & \frac{2g+1}{2g+2} \\
\hline
\text{dimension} & g & g+1 & g+2 \\
\end{array}
\]

Let \( \{f_0, \ldots, f_g+1\} \subset \mathbb{C} \) be a basis, and define for \( m \in \mathbb{P}^g \)

\[(6.17) \quad q(m) = [f_0(m) : \ldots : f_{g+1}(m)] \in \mathbb{P}^{g+1}. \]

For \( m = p \) this is unsuitable, since (except for constants) the function will blow up. Writing \( \tilde{z} \) for a local holo. coord. at \( p \), set

\[(6.18) \quad \tilde{q}(p) = [(z_{\tilde{f}_0}(p) : \ldots : z_{\tilde{f}_{g+1}}(p)) \in \mathbb{P}^{g+1}. \]

We already have \( \lim_{m \to p} q(c(m)) = q(p) \) and have constructed an analytic map between complex manifolds \( M \) and \( \mathbb{P}^{g+1} \), provided

\[(6.17-18) \] do not yield \([0; \ldots; 0]\) at any point of \( M \).

To check this doesn't happen, and that \( q \) is injective,
we use (6.16):

- for \( q \neq p \), \( L_q \nsubseteq L \Rightarrow \exists f \in L \) not vanishing at \( q \)
  \[ \Rightarrow \) not all \( f_i(q) = 0 \]

- \( L_p \nsubseteq L \Rightarrow \exists f \in L \) with \( \nu_p(f) = -(2g+1) \Rightarrow \) not all \( (\varepsilon^{2g+1}_p f)(p) = 0 \)

- for \( p, q, r \) distinct \( L_q, r \nsubseteq L_p \Rightarrow \exists f \in L \) vanishing at \( q \) but not at \( r \)
  \[ \Rightarrow \) \( \phi(q) \neq \phi(r) \)

- for \( q \neq p \), \( L_q, r \nsubseteq L_p \Rightarrow \exists f \in L \) vanishing at \( q \) but with \( (\varepsilon^{2g+1}_p f)(p) \)
  \[ \Rightarrow \) vanishing at \( p \Rightarrow \phi(q) \neq \phi(p) \).

\[ \Box \]

Remark 3: (i) This can be refined further to show \( M \subseteq \mathbb{P}^3 \) and \( M \hookrightarrow \mathbb{P}^2 \) with only “normal crossings” or “000” singularities (locally of form \( xy=0 \))

(ii) One can also use a blow up for \( M \) to set an embedding in \( \mathbb{P}^{g+1} \) (“canonical curve”), but this doesn’t work for hyperelliptic \( RS \text{'s} \) — those with a degree-2 map to \( \mathbb{P}^1 \).

(iii) In fact, we can even check that the image of the above mapping \( M \to \mathbb{P}^{g+1} \) will be smooth:

\[ L_q, r \nsubseteq L_p \Rightarrow \exists f \in L_q \) vanishing to exactly 1st order at \( q \)
  \[ \Rightarrow \) derivative of \( \phi \) (in local coords.) is nonzero true
  + similar check at \( p \).

Before turning to periods and the Riemann bilinear relations, here is one more important application of sheaf technology. Let

\[ \tau: M \to N \]

be a surjective mapping of \( CRS \text{'s} \) (most commonly \( N \) will be \( \mathbb{P}^1 \)) of degree \( d \) \( (\text{:= cardinality of } \tau^{-1}(q) \text{ for generic } N \) . Let
\[ \Delta = \{ p_1, \ldots, p_m \} \subseteq M \] denote the set of ramification points, of degree \( r_i \; \text{(i.e., locally the map looks like } z \mapsto z^{r_i} \text{)} \); we have
\[ \pi(\Delta) = \{ q \in N \mid |\pi^{-1}(q)| < d_\pi \} \]. Define
the ramification divisor
\[ \{ \begin{align*}
R_\pi &= \sum (r_i - 1) [p_i] \in \text{Div}(M) \\
r_\pi &= \deg R_\pi
\end{align*} \]
Consider \( \omega \in K^1(M)^{\geq 0} \), with \( (\omega) \cap \pi(\Delta) = 0 \). Then under \( z \mapsto z^{r_i} = w \), \( dw \) pulls back to \( r_i z^{r_i - 1} dz \), and so we have for \( \pi^* \omega \in K^1(M) \)
\[ (\pi^* \omega) = (\pi^* (\omega)) + R_\pi \]
\[ \implies g_M - 2 = \deg (\pi^* \omega) = \deg (\pi^* (\omega)) + \deg R_\pi = d_\pi \cdot \deg (\omega) + r_\pi \]
giving the
Riemann-Hurwitz formula:
\[ g_M = \frac{d_\pi \cdot (g_M - 1)}{2} + 1. \]

**Example 2:** For a hyperelliptic \( RS \) \( M \xrightarrow{2:1} \mathbb{P}^1 \),
with 2b branch pts., \( g_M = b - 1 \).

**Note:** the restrictions imposed by this formula on the possible data involved: we must have
\[ \left\lfloor \frac{2}{r_\pi} \right\rfloor \] 
\[ d_\pi \mid g_M - \frac{r_\pi}{2} - 1 \]
Let \( \{ w_1, \ldots, w_g \} \subset SL^1(M) \) be a basis.

\[\{ y_1, \ldots, y_{2g} \} \subset H_1(M, \mathbb{Z}) \] the symplectic basis described above.

Then we have the period matrix \( \Pi_j = \begin{pmatrix} f_j & \omega_j \\ \omega_j^T & 0 \end{pmatrix} \in \mathbb{C}^{2g} \) and the period vector \( \pi_j \).

Proposition 2: The \( e_j \) are \( \mathbb{R} \)-linearly independent (viewed as vectors in \( \mathbb{R}^{2g} \)).

Proof: If \( 0 = \Pi \alpha \) (\( \alpha \in \mathbb{R}^{2g} \)) then \( 0 = \left( \begin{array}{c} 1 \\ \Pi \end{array} \right) \alpha \Rightarrow \)

\[ \text{rank} \begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix} < 2g \Rightarrow \exists b \in \mathbb{C}^{2g} \setminus \{0\} \text{ s.t. } b \left( \begin{pmatrix} \Pi & 0 \\ 0 & \Pi \end{pmatrix} \right) = 0 \Rightarrow \]

\[ \sum_{i=1}^{2g} b_i a_i \omega_i = 0 \Rightarrow [\omega | \phi] = 0 \Rightarrow \omega | \phi = 0 \quad \square \]

So \( \Lambda_M := \mathbb{Z} \langle \bar{\tau}_1, \ldots, \bar{\tau}_g \rangle \subset \mathbb{C}^g \) is a full lattice, or more intrinsically \( H_1(M, \mathbb{Z}) \subset SL(M)^\vee \).

Definition 2: The Jacobian of \( M \) is the complex \( g \)-torus

\[ J(M) := \frac{SL^1(M)}{H_1(M, \mathbb{Z})} \cong \mathbb{C}^g \]

\[ \uparrow \Lambda_M \]

equivariant against the basis \( \{ w_1, \ldots, w_g \} \).
Now for any \( \psi \in \mathcal{H}(M) \otimes \mathcal{H}(M) \), we have the equality of functionals on \( \mathcal{H} \):

\[
(6.19) \quad [\psi] = \sum_{j=1}^{g} \left( \pi_j(\psi) \left[ \psi_{j^{-1}} \right] - \pi_{j^{-1}}(\psi) \left[ \psi_j \right] \right)
\]

where we are using the equivalent pairings \( \langle , \rangle \) from (6.4). For \( \omega, \psi \in \mathcal{H}(M) \)

\[
(6.20) \quad 0 = \int_{\mathcal{M}} \omega \wedge \psi = \left[ [\omega], [\psi] \right] = \sum_{j=1}^{g} \left( \pi_j(\psi) \pi_{j^{-1}}(\omega) - \pi_{j^{-1}}(\psi) \pi_j(\omega) \right)
\]

\[
(6.21) \quad 0 < i \int_{\mathcal{M}} [\omega, \bar{\omega}] = i \left[ [\omega], [\bar{\omega}] \right] = -i \sum_{j=1}^{g} \left( \bar{\pi}_j(\omega) \pi_{j^{-1}}(\omega) - \bar{\pi}_{j^{-1}}(\omega) \pi_j(\omega) \right)
\]

We can restate the result in matrix form:

\[
(6.22) \quad \begin{cases}
(\text{i}) & TT^*T = 0 \\
(\text{ii}) & \sqrt{-1} TT^*T > 0
\end{cases}
\]

Writing \( T = \begin{pmatrix} A & B \\ G & S \end{pmatrix} \), (6.22) becomes

\[
(6.22) \quad \begin{cases}
(\text{i}) & A^TB - B^TA = 0 \\
(\text{ii}) & \sqrt{-1} (A^TB - B^TA) > 0
\end{cases}
\]

In particular, \( \sqrt{-1} T \mathcal{T}^*T (A^TB - B^TA) \mathcal{T} > 0 \) \( \forall \mathcal{T} \neq 0 \Rightarrow A \) cannot have nontrivial left-inverse \( A \) is invertible.

\* e.g., \( \mathcal{C} = TT^*T = \begin{pmatrix} \pi^T & \pi^T \\ G & S \end{pmatrix} \begin{pmatrix} 0 & E_g \\ E_g & 0 \end{pmatrix} \begin{pmatrix} \pi & \pi \\ G & S \end{pmatrix} = \begin{pmatrix} \pi^T & \pi^T \\ G & S \end{pmatrix} \begin{pmatrix} \pi & \pi \\ G & S \end{pmatrix} (\text{matrix of the product comm. to } (6.20) \text{ w/ substitute } \omega, \psi) \)
Using $A^{-1}$ to change our $w$-boots, we have

$$\mathbf{T}' = A^{-1} \mathbf{T} = \begin{pmatrix} I_g & \frac{A^{-1} \mathbf{B}}{z} \\ \frac{z}{z} & z \end{pmatrix} \quad \Rightarrow \quad (6.23)$$

with $A = I_g$, $B = z$

\[z^* \mathbf{T} z = 0\]

\[\sqrt{-1} \left( \frac{z^* \mathbf{T} z}{z} \right) > 0\]

and hence the

**Theorem 2:**

**Ricci-Čech Bilinear Relations:**

1. $z = z^*$
2. $\text{Im}(z) > 0$

---

**Complex tori**

Now we shall approach these "bilinear relations" from within a different (but related) context. Let $\Lambda \subset V \cong \mathbb{C}^n$ be a full lattice, $\mathcal{T} := V/\Lambda$ the complex $n$-torus. Clearly $V \cong \mathcal{T}$, and we also write $W := V^\mathbb{C}$. There are 2 natural choices of basis for $W_\mathbb{C} (= T_{\mathbb{C},0}^\mathbb{R})$:

- given a basis $\lambda_1, \ldots, \lambda_{2n}$ of $\Lambda$ (hence of $W \oplus W_\mathbb{C}$),
  - let $\mathcal{B} := \{ dx_1, \ldots, dx_{2n} \}$ denote the dual basis of $W_\mathbb{C}$.

* viz., \[\mathbf{T} = \begin{pmatrix} \omega_i & \sigma_i \\ \sigma_i & \omega_i \end{pmatrix} \Rightarrow A^{-1} \mathbf{T} = A^{-1} \begin{pmatrix} \omega_i & \sigma_i \\ \sigma_i & \omega_i \end{pmatrix} = \begin{pmatrix} \omega_i \sigma_i \\ \sigma_i \omega_i \end{pmatrix} \]
\[ W^0 \cong W^+_0 \oplus W^-_0 \ (\cong V \oplus \overline{V}), \quad \text{where} \ V \xrightarrow{\phi} W^+_0 \text{ is some basis for } W^+_0 \xrightarrow{n \mapsto n - i \ I(n)} \text{ co-spin.} \]

One can obviously extend these to translation-invariant differential forms on \( \mathcal{T} \), and this leads to isomorphism (as in problem set 1 Ex. 6 solution)

\[ \Lambda^k W^\vee_0 \cong H^{dr}_{\mathcal{T}}(\mathcal{T}, \mathbb{C}) \]

for each \( k \). In fact, using the \((p, q)\) decomposition of the left-hand side \( \cong \bigoplus_{p+q=k} \Lambda^p W^+_0 \otimes \Lambda^q W^-_0 \) we set the

**Proposition 3:** (Hodge decomposition)

\[ H^k_{dr}(\mathcal{T}, \mathbb{C}) \cong \bigoplus_{p+q=k} \mathbb{C} \langle \left\{ \frac{d\omega_I \wedge d\overline{\omega}_J}{\left| I,J \right|^2} \right\} \rangle \]

\[ \text{"} H^{p,q}(\mathcal{T}) \text{"} \]

It is also clear that we have

\[ H^k(\mathcal{T}, \mathbb{Z}) \cong \mathbb{Z} \langle \left\{ dx^k \right\} \rangle, \quad k < n. \]

Recall that a necessary condition for \( \mathcal{T} \) to possess a
projective embedding, is for it to have a Kähler metric whose Kähler class is integral — or equivalently, a d-dual (using transluscent invariant) positive (1,1)-form with integral class in $H^2$.

This is called a polarizing form, and we look for conditions (essentially on $\Lambda$) under which one exists.

Start by identifying

$$[Y_i]_i \in \Lambda$$

where we think of $Y_i$ as a straight segment in $V$ from $0$ to $i\epsilon \Lambda$, and $d\epsilon_1$ as differentials of coordinates on $V \cong \mathbb{C}^n$. In fact, with these identifications we have

$$\mathcal{T} (= \frac{V}{\Lambda}) \cong \frac{\mathbb{R}^n(T) \setminus \Lambda}{H_1(T, \mathbb{Z})}$$

Define the period matrix

$$\Pi = \{\pi_{ij} = \int_{Y_j} d\epsilon_i\}$$

so that the change-of-basis matrix

$$n^\ast [id] = \left(\begin{array}{c|c} \Pi & 0 \\ \hline 0 & \end{array}\right) = \tilde{\Pi}$$

with inverse

$$[id]_n = \Pi^{-1} \left(\begin{array}{c|c} \Xi & 0 \\ \hline 0 & \end{array}\right)$$

so that

$$d\epsilon_i = \frac{\partial}{\partial \epsilon_i}$$

and

$$d\epsilon_i = \frac{\partial}{\partial \epsilon_i}$$
\[ dx_j = \sum \delta_{ij} \, dx_i + \sum \delta_{ij} \, dx_j \quad \text{or}, \quad \text{writing} \quad dx = \left( \begin{array}{c} dx_1 \\ \vdots \\ dx_n \end{array} \right) \quad \text{and} \]
\[ \delta^i_j = \left( \begin{array}{c} \delta^i_1 \\ \vdots \\ \delta^i_n \end{array} \right) = \left( \begin{array}{c} \delta^i_1 \\ \vdots \\ \delta^i_n \end{array} \right), \quad \prod \delta^{-1} \, dx = dx \]

Let \( \omega \) be a translation - invariant, \( d \)-closed
2-form with integral cohomology class: so
\[ \omega = \frac{1}{2} \sum q_{ij} \, dx_i \wedge dx_j \quad (q_{ij} \in \mathbb{Q} \ n \times n \text{ skew-symmetric} \text{ matrix}) \]

We will prove the linear algebra

**Lemma 1**: We may choose the \( x_1' / \ldots / x_n' / dx \) - basis so that
\[ Q = Q_{\Delta} = \left( \begin{array}{cc} 0 & \Delta \\ -\Delta & 0 \end{array} \right), \]
where
\[ \Delta = \left( \begin{array}{ccc} \delta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_n \end{array} \right) \]
and \( \delta_1, \ldots, \delta_n (\in \mathbb{N}) \) are invariants of \( Q \).

If \( as \rightarrow 0 \) then (by corollary E.2) \( \omega^n \neq 0 \Rightarrow \) all \( \delta_i \neq 0 \).
Assume this linear form.

Now then
\[ \omega = \frac{1}{2} \, dx \wedge Q \, dx = \frac{1}{2} \, \prod \, dx \wedge \prod \, Q \, \prod \, \Delta \]
\[ = \frac{1}{2} \left( \prod dx_i \wedge dx_j \right) \left( \begin{array}{c|c} \prod Q_{ij} & \prod Q_{ij} \\ \hline \prod Q_{ij} & \prod Q_{ij} \end{array} \right) \left( \begin{array}{c} dx_i \\ dx_j \end{array} \right) \]

\[ \text{Clearly, calling this} \]
\[ a = -a \Rightarrow \]
\[ a = -a \Rightarrow \]
\[ a = -a \Rightarrow \]
and \( \omega \) is \( \{ \text{of type } (1,1) \text{ and } \sigma > 0 \} \)

\[
\text{pf of } \text{Thm E.1} \quad \text{with } \eta > 0
\]

\[
\Longrightarrow \quad \tau_{\hat{\Pi}} Q^{-1} \hat{\Pi}^{-1} = i \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}
\]

\[
\text{take inverse } \eta = \eta^{-1}
\]

\[\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \quad \text{with } \eta > 0.
\]

Writing \( \hat{\Pi} = (A \ B) \) as in the CRS setting, we get

\[
i^* \tau_{\hat{\Pi}} Q^{-1} \hat{\Pi} = i \begin{pmatrix} A & B \\ \tilde{A} & \tilde{B} \end{pmatrix} \begin{pmatrix} 0 & -\Delta_\sigma^{-1} \\ \Delta_\sigma & 0 \end{pmatrix} \begin{pmatrix} \tilde{t} A & \tilde{t} \tilde{A} \\ \tilde{t} B & \tilde{t} \tilde{B} \end{pmatrix}
\]

\[
= i \begin{pmatrix} B \Delta_\sigma^{-1} A - A \Delta_\sigma^{-1} \tilde{B} & \mathbf{X} \\ B \Delta_\sigma^{-1} \tilde{A} - A \Delta_\sigma^{-1} \tilde{B} & \mathbf{X} \end{pmatrix}
\]

So that \( \eta > 0 \iff i \begin{pmatrix} B \Delta_\sigma^{-1} A - A \Delta_\sigma^{-1} \tilde{B} \end{pmatrix} > 0 \Rightarrow \text{ A invariant.}
\]

Assume this happens.

\[
\Rightarrow \text{ write with } \hat{\Pi} = (\Delta_\sigma \ Z).
\]

Hence, our 2 conditions on \( \omega \) become \( Z \Delta_\sigma^{-1} \Delta_\sigma - \Delta_\sigma \Delta_\sigma^{-1} \tilde{Z} = 0 \)

\( \Rightarrow \tilde{Z} = Z \), and \( i(\bar{Z} - Z) > 0 \). We conclude.

**Theorem 3**: \( \hat{\Pi} \) admits a polarizing form \( \Leftrightarrow \)

\[
\exists \text{ basis for } H_1(T; \mathbb{Z}) (\cong \Lambda) \text{ and } \mathcal{S}'(T) (\cong V^*) \text{ such that }
\]

the period matrix \( \hat{\Pi} \) is of the form \( (\Delta_\sigma \ Z) \) with

\[
\text{Im } \bar{Z} > 0 \quad \text{AND} \quad \bar{Z} Z = Z.
\]
Remark 4: (i) There are the conditions under which \( T \) admits a projective embedding, i.e., is an abelian variety (and usually denoted \( A \)). Clearly, the general one is not.

(ii) There are 2 ways to embed it in \( \mathbb{P}^r \) in this case:
- Kodaira’s theorem, or explicit construction of theta functions.

If all \( d_i = 1 \), then \( \tau \) is called a principal polarization of \( T \).

Given a CRS \( M \), then its Jacobian \( J(M) \cong \Omega^1(M) / H_1(M, \mathbb{Z}) \) is polarized is a tautology; in fact, by taking \( \Omega^1(M) / H_1(M, \mathbb{Z}) \) it is principally polarized.

Moduli

The \( g \)-th Siegel upper half-space

\[ \mathbb{H}_g := \left\{ Z \in \mathbb{M}_{g \times g}(\mathbb{C}) \mid \tau Z = Z \quad \text{and} \quad \text{Im}(Z) > 0 \right\} \]

parametrizes the set of principally polarized abelian varieties (PPAV's). Over \( \tau \in \mathbb{H}_g \), we have \( \mathbb{A}_\tau = \mathbb{C}^g / \Lambda_\tau \) where \( \Lambda_\tau = \mathbb{Z} \langle \text{column } \tau \rangle \).

In order that each isomorphism class of PPAV occur any one, we take the quotient by \( \mathbb{P}_g := \text{Sp}_{2g}(\mathbb{Z}) \)

\( \mathbb{H}_g \) is just the familiar upper half-plane. For \( g=1 \), any \( \tau \) (i.e., \( \tau \in i \mathbb{R} \) is a lattice) is in \( \pm \mathbb{H}_2 \), so all complex tori are algebraic. (False for \( g \geq 3 \)).
\[ A_g := \Gamma_g \backslash \mathbb{H} \]

where \( g = (A \, B) \) acts by \( g(z) := (Az + B)(Cz + D)^{-1} \). This action on the "period point" is equivalent to a symplectic-linear change of integral basis; this preserves the polarizing form.

Remark 5: The genuine miracle here is that \( A_g \) has a projective embedding and is therefore an algebraic variety, called the \( g \)th Siegel modular variety. The embedding is given by Siegel modular forms (more on these later).

What about moduli of curves? First some more general discussion.

Kodaira & Spencer developed the theory of deformations of complex structures on manifolds. Since manifolds are made up of charts, a "deformation" should be a shifting of the chart (on the overlaps). The tangent to a deformation should therefore identify with a class in \( H^1(M, \Theta'_M) \), where \( \Theta'_M := \Theta(T^{(1,0)}_M) \) is the sheaf of holomorphic vector fields.

To be more precise, suppose \( \pi : X \to \mathbb{D} \) is a submersive holomorphic map from a complex \( n \)-manifold onto the unit disk, with compact fibers. Then the \( \{X_t := \pi^{-1}(t)\}_{t \in \mathbb{D}} \) are a family of
compact complex \((n-1)\)-manifolds, all diffeomorphic \(*\) but with varying complex structure. Considering the short exact sequence of sheaves on \(M := X_0\)

\[
0 \to \Theta^1 \to \Theta^1 | _{X_0} \to \left( \pi^* T_{D, \phi_0}^{(1,0)} \right) \otimes \Theta^0_M \to 0
\]

the image of \(\frac{1}{2} \delta t \otimes 1\) under the connecting homomorphism

\[
H^0(M, (\pi^* T_{D, \phi_0}^{(1,0)}) \otimes \Theta^0_M) \to H^1(M, \Theta^1_M)
\]

defines the Kodaira–Spencer class. This will be used later in the course.

Now consider the case where \(M\) is a CRS of genus \(g\).

I ask you to believe, for the next proof, that \(H^1(M, \Theta^1_M)\) is the "tangent space to \(M\) (moduli space of genus \(g\) CRS's) at \(M\)."

For a proof of this theorem without such an assumption, see p. 292 of my Alg. geom. book online.

Theorem 4 (Riemann): Compact surfaces of genus \(g \geq 2\) have

\(\text{up to isomorphism} \) \(3g - 3\) moduli. So

\[
\dim(M_g) = \begin{cases} 
3g - 3 & g \geq 2 \\
1 & g = 1 \\
0 & g = 0
\end{cases}
\]

\(*\) This is a consequence of Fubini's theorem. Any \(C^\infty\) lifting of the vector field \(\partial_\phi\) on \(D\) to \(X\) defines a rank-one, hence integrable, distribution and thus an isomorphism \(X \cong M \times D\) of \(C^\infty\) manifolds. (It does \(\neq\) respect the complex structures \(\).)
Proof: \[ H'(\theta'_m) = (H'(\theta'_m))^{\psi} = H'(\mathcal{O}(2K)) \cong H^0(\mathcal{O}(2K))^\vee \] (numerical divisor)

The dimension of this space is

for \( g \geq 2 \):
\[
\tilde{h}(-K) = g - \deg(-K) - 1 = 3g - 3
\]

\[ (\deg(K) = 2g - 2 \leq 0 \text{ for } g \geq 2) \]

\[ \Rightarrow \tilde{h}(K) = 0 \]

for \( g = 1 \):
\[ \tilde{h}(O) \Rightarrow \dim = 1 \]
(or \((\mathcal{M}, \mathcal{T})\), introduced by "de"

for \( g = 0 \):
\[ \deg K < 0 \Rightarrow \dim = 0. \square \]

Corollary 7: A general CRS of genus \( g \geq 3 \) is non-hyperelliptic.

Proof: Hyperelliptic CRS's are determined by the \( 2g+2 \) branch points of their 2:1 mapping to \( \mathbb{P}^1 \), modulo the action of \( \text{PGL}_2(\mathbb{C}) \) which can send any 3 of these branch points to 0, 1, \( \infty \). So they have \( 2g-1 \) moduli, which is less than \( 3g-3 \) for \( g \geq 3 \). \square

Corollary 8: For \( g \geq 4 \), the genus \( g \)-dimensional PPAV is not the Jacobian of a CRS.

Proof: First note that
\[ \dim (\mathcal{A}_g) = \dim (\text{space of symmetric g symmetric matrices}) = \binom{g+1}{2}. \]
New look at the table

<table>
<thead>
<tr>
<th>$g$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>dim $M_g$</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>...</td>
</tr>
<tr>
<td>dim $J_g$</td>
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<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>...</td>
</tr>
</tbody>
</table>

Certainly, $\dim M_g \geq \dim J_g(M_g)$, and so we are done.

In fact, the Jacobian mapping is an embedding — this is called the **Torelli theorem**, and we’ll eventually prove it using Hodge theory later. The **Schofflag problem** is to describe the Jacobian locus (image of $J$) in $A_g$. It is still open for $g \geq 5$. 

\[
\text{Jacobian mapping}
\]