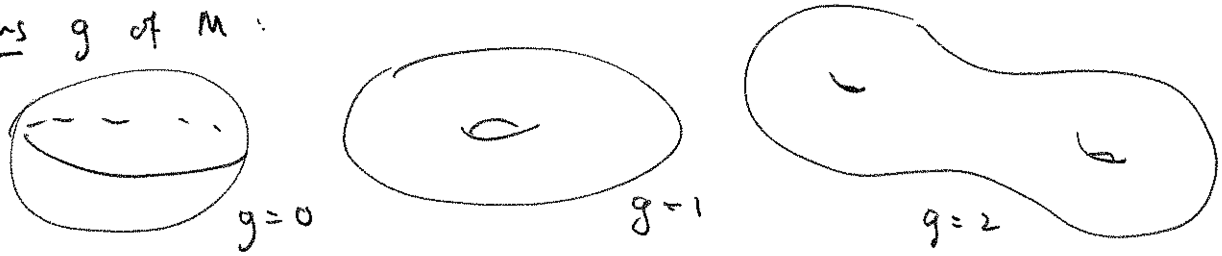
G. Riemann surfaces and complex tori

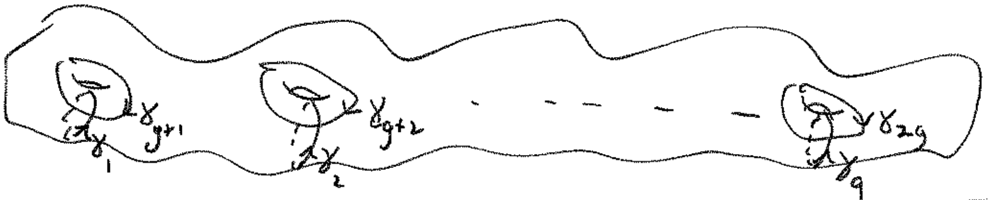
We now study some applications of sheaf cohomology and the de Rham theorem in a classical setting, partly to have some concrete examples of Hodge decompositions & bilinear relations before we meet them in greater generality.

As defined above, a *compact Riemann surface* (CRS) M → is just a compact complex 1-manifold. M is Kähler, of course, since $d\omega = 0$ automatically (no 3-forms on M). We will see below that it is in fact projective.

The isomorphism class of M as a C^∞ manifold is given by the *genus* g of M:



and this determines its homology $H_1(M,\mathbb{Z}) \cong \mathbb{Z}\langle \gamma_1,\dots,\gamma_{2g}\rangle$:



Note the (perfect) intersection pairing

$$(G.1) \qquad H_1(M,\mathbb{Z}) \times H_1(M,\mathbb{Z}) \longrightarrow \mathbb{Z},$$

with matrix

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Aside: Note that the subgroup of $GL_{2g}(\mathbb{Z})$ defined (w.r.t. basis $\{\gamma_j\}$, if you will) by

$$\boxed{{}^tMJM = J}$$

is called the *symplectic group* $Sp_{2g}(\mathbb{Z})$.

As for the cohomology, we have the cup (wedge) product

(6.2) $$H^1_{dR}(M,\mathbb{C}) \times H^1_{dR}(M,\mathbb{C}) \xrightarrow{\cup} H^2_{dR}(M,\mathbb{C}) \xrightarrow[\cong]{\int_M} \mathbb{C}$$

as well as the (perfect) pairing induced by integration

(6.3) $$H^1_{dR}(M,\mathbb{C}) \times H_1(M,\mathbb{C}) \xrightarrow{\int} \mathbb{C} .$$

We get a composite isomorphism

$$\begin{tikzcd} H_1(M,\mathbb{C}) \arrow[r, "(6.1)", "\cong"'] & H^1(M,\mathbb{C}) \arrow[r, "(6.3)", "\cong"'] & H^1_{dR}(M,\mathbb{C}) \\ {[\gamma]} \arrow[rr, mapsto, "\eta"] & & {[\eta_\gamma]} \end{tikzcd}$$

any d-closed 1-form with the right class (but see below)

which begs the question: are the pairings all compatible, viz., does

(6.4) $$\begin{tikzcd} H^1_{dR}(M,\mathbb{C}) & \times & H^1_{dR}(M,\mathbb{C}) \arrow[dr, "\cup"] & \\ H_1(M,\mathbb{Z}) \arrow[u, "\eta"] & \times & H^1_{dR}(M,\mathbb{C}) \arrow[u, equal] \arrow[r, "\int"] & \mathbb{C} \\ H_1(M,\mathbb{Z}) \arrow[u, equal] & \times & H_1(M,\mathbb{Z}) \arrow[u, "\eta"] \arrow[ur, "\cap"] & \end{tikzcd}$$

commute?

In fact, the only issue here is the "top triangle", that is, do we have

$$\int_M \eta_\gamma \wedge \eta_{\gamma'} = [\gamma] \cap [\gamma'] \quad ?$$

There is a simple construction for $\eta_\gamma \in A^1(M)_{d\text{-closed}}$ which makes this obvious: take a C^∞ embedding of $|\gamma_i| \times \Delta$ in a small tubular neighborhood of $|\gamma_i|$, which gives a diagram

$$\begin{tikzcd} {|\gamma_i| \times \Delta} \arrow[r, hook, "\iota"] \arrow[d, "\pi"] & M \\ \Delta & \end{tikzcd}$$

(Δ = unit disk)

Taking a C^∞ bump 1-form η on Δ with $\int_\Delta \eta = 1$, we set $\eta_\gamma := \iota_* \pi^* \eta$. By Fubini, one sees that at intersections

$$\int_{\substack{\text{nbhd}\\ \text{of intersection}\\ \text{of } \gamma_i \,\&\, \gamma_j}} \eta_{\gamma_i} \wedge \eta_{\gamma_j} = \pm \int_{\Delta\times\Delta} \pi_1^* \eta \wedge \pi_2^* \eta = \pm \left(\int_\Delta \pi^* \eta \right)^2 = \pm 1 ,$$

(according to "right-hand rule")

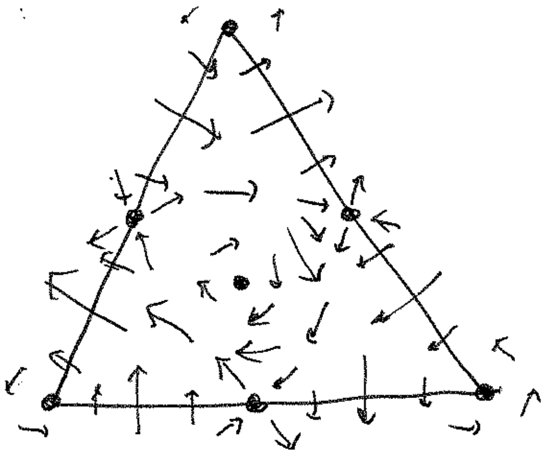
which $\Longrightarrow$ (6.4) commutes (and (6.2) is perfect).

So periods of C^∞ 1-forms are whatever we want them to be. Holomorphic forms are more interesting. To see this we will require the following "preliminary" result:

<u>Poincaré-Hopf Theorem</u>: (i) Given $\vec{v} \in C^\infty(M, T_M)$, $\sum_{p\in M} \text{ind}_p \vec{v} = \chi_M$. *

(ii) Given $\omega \in \underset{\text{meromorphic 1-forms}}{\mathcal{K}^1(M)} \setminus \{0\}$, $\left(\underset{[\text{w/mult.}]}{\#\text{ of } 0\text{'s of } \omega}\right) - \left(\underset{[\text{w/mult}]}{\#\text{ of poles of } \omega}\right) = 2g-2$.

<u>Proof</u>: (i) "triangulate" M, and draw the following $\overset{C^\infty}{\text{vector}}$ field on
(sketch) each triangle:



which clearly glue together to give a global vector field on M, with indices -1 at the marked points on the edges and $+1$ at the marked points on the faces + vertices. Hence,

$$(6.5) \quad \sum \text{ind}_p \vec{v} = \#F - \#E + \#V = \chi_M = 2-2g .$$

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* here $\text{ind}_p \vec{v}$ is the # of anti-clockwise rotations done by the head of $\vec{v}$ as one goes around a small circle about p (once, anti-clockwise). It can be nonzero only if $\vec{v}(p) = 0$.

It's fairly easy to see that this invariant is continuous under C^∞ deformation, hence that (G.5) holds for <u>any</u> vector field $\vec{v}$. It still holds if we allow $\vec{v}$ to have singularities at a finite number of points $\{p_1, \dots, p_n\}$ (i.e. $\vec{v} \in C^\infty(M \setminus \{p_1,\dots,p_n\}, T_M)$) provided one adds in the indices of $\vec{v}$ at the p_i into the sum.

(ii) (G.5) even holds if $\vec{v}$ is replaced by a smooth 1-form $\eta \in A^1_{\mathbb{R}}(M \setminus \{p_1,\dots,p_n\})$, by using a metric g to identify $\mathbb{T}M \underset{C^\infty}{\cong} \check{T}_M$. The corresponding notion of index, if $\eta \underset{\substack{\text{loc} \\ (\text{coords.} \\ \text{at } p)}}{=} F\,dx + G\,dy$, is

$$\text{(G.6)} \qquad \text{ind}_p \eta := \frac{1}{2\pi} \oint d \arctan\left(\frac{G}{F}\right),$$

and once again the sum in (G.5) must be over all 0's & poles of η.

If $\omega \in K^1(M)$ has local form $\omega \underset{\text{loc}}{=} f\,dx + g\,dy$ (f, g cx. valued), then $\eta := \text{Re}(\omega) \underset{\text{loc}}{=} \text{Re}(f)\,dx + \text{Re}(g)\,dy$. Let $p =$ "0 or pole" of ω, and put $\nu_p := \text{ord}_p(\omega)$. In a local <u>holomorphic</u> coord. system at p,

$$\omega \overset{\text{loc}}{\approx}{}^{*} z^\nu dz = r^\nu(\cos\nu\theta + i\sin\nu\theta)(dx + i\,dy)$$

$$= r^\nu(\cos(-\nu\theta) - i\sin(-\nu\theta))\,dx + r^\nu(\sin(-\nu\theta) + i\cos(-\nu\theta))\,dy.$$

For the real part, then,

$$\frac{\eta}{r^\nu} \approx \cos(-\nu\theta)\,dx + \sin(-\nu\theta)\,dy$$

and so by (G.6)

$$\text{ind}_p \eta = \frac{1}{2\pi} \oint d[-\nu\theta] = -\nu_p$$

$$\Rightarrow \sum_{p \in M} \nu_p = 2g - 2.$$

$\square$

The comparable result for a meromorphic function $f \in \mathcal{M}(M)\setminus\{0\}$ is

$$\text{(G.6)} \quad \boxed{\underbrace{\overline{\begin{pmatrix} \#\, 0\text{'s of } f \\ \text{w/mult.} \end{pmatrix} - \begin{pmatrix} \#\text{ poles of } f \\ \text{w/mult.} \end{pmatrix}}}_{} = \sum \text{Res}_p \frac{df}{f} = \frac{-1}{2\pi i} \int_{\partial(M \setminus \cup D_\epsilon(p_j))} df/f \overset{\text{Stokes}}{=} \frac{-1}{2\pi i} \int_{M\setminus\cdots} \cancel{d(df/f)}^{\,0} = 0}.$$

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* up to multiplication by a loc. holomorphic fcn., which will not affect index.

We can rephrase these statements in terms of the group of (Weil) <u>divisors</u>

$$\begin{tikzcd}
\mathrm{Div}(M) := \Big\{ \arrow[d, "\deg\ (= \text{degree homomorphism})"] & \overset{\text{``}=: D\text{''}}{\displaystyle\sum_{p\in M} d_p[p]} \arrow[d, mapsto] & \Big| \; d_p \in \mathbb{Z},\ \neq 0 \text{ for only finitely many } p \Big\} \\
\mathbb{Z} \ni & \sum d_p =: \text{``}d\text{''} &
\end{tikzcd}$$

Write $\mathrm{Div}(M)^0 := \ker(\deg.)$

- D is <u>effective</u> $\Leftrightarrow$ $d_p \geq 0$ $(\forall p)$ $\Leftrightarrow$ "$D \geq 0$"
- $D \geq E$ $\Leftrightarrow$ $D - E \geq 0$
- a given D may be written $D_+ - D_-$, with $D_+, D_- \geq 0$.

<u>Example 1</u>: (i) $(f) := \sum_{p\in M} \nu_p(f)[p]$ induces a group homomorphism

$$(\cdot) : \mathcal{M}(M)^* \longrightarrow \mathrm{Div}(M)^0$$

(ii) Given $\omega \in K^1(M)^*$, writing $\omega = f\,dz$ in local coordinates one defines $\nu_p(\omega) := \nu_p(f)$; this yields $(\omega) := \sum_{p\in M} \nu_p(\omega)[p]$, and Poincaré-Hopf $\Longrightarrow$

$$\deg((\omega)) = 2g-2.$$

□

We now have the crucial

<u>Definition 1</u>: $\mathcal{O}(D) :=$ sheaf of meromorphic functions f $\underset{\text{(locally)}}{\underline{\text{satisfying } (f) + D}} \geq 0$ (or $f \equiv 0$)

$\Omega^1(D) :=$ sheaf of mero. 1-forms satisfying (locally)
$(\omega) + D \geq 0$ (or $\omega \equiv 0$)

$\mathcal{L}(D) := \mathcal{O}(D)(M)$ $\left(= H^0(\mathcal{O}(D))\right)$, $\mathcal{I}(D) := H^1(\mathcal{O}(D))$

(the "M" is understood)

$l(D) := \dim \mathcal{L}(D)$, $i(D) := \dim \mathcal{I}(D)$.]

For $D \geq 0$, one has short-exact sequences

(G.7) $$0 \to \mathcal{O} \to \mathcal{O}(D) \to \underbrace{\bigoplus \iota^p_* \mathbb{C}^{d_p}}_{\text{records principal part}} \to 0$$

(G.8) $$0 \to \Omega^1 \to \Omega^1(D) \to \bigoplus \iota^p_* \mathbb{C}^{d_p} \to 0$$

with associated long-exact sequences

(G.9) $$0 \to \underset{\mathbb{C}}{\underset{\wr\wr}{\mathcal{O}(M)}} \to \mathcal{L}(D) \to \mathbb{C}^{d\,(=\deg D)} \to H^1(\mathcal{O}) \to \mathcal{I}(D) \to 0$$

(G.10) $$0 \to \Omega^1(M) \to \Omega^1(D)(M) \to \mathbb{C}^d \to H^1(\Omega^1) \to H^1(\Omega^1(D)) \to 0$$

in which the alternating sum of dimensions must be zero (<u>exercise</u>).

For $D > 0$ we also have

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \bigoplus \iota^p_* \mathbb{C}^{d_p} \to 0 \quad \implies$$

(G.11) $$\underset{(\mathcal{O}(-D)(M))}{\underset{\parallel}{0}} \to \underset{\mathbb{C}}{\underset{\wr\wr}{\mathcal{O}(M)}} \to \mathbb{C}^d \to \mathcal{I}(-D) \to H^1(\mathcal{O}) \to 0,$$

(holo. fns. vanishing somewhere are 0)

and so for $d >> 0$

$$(G.12) \quad \begin{cases} \ell(D) \geq d + O(1) & \text{(in the sense of asymptotics: i.e., something bounded)} \\ \dim \Omega^1(D)(M) \geq d + O(1) \\ i(-D) = d + O(1) \end{cases}$$

This immediately gives

Proposition 1 : Noncontant meromorphic functions and nonzero meromorphic forms exist. //

Let $\omega \in K^1(M)^*$ then, and set

$$K := (\omega) \in \mathrm{Div}(M) \; ;$$

this canonical divisor is well-defined modulo $(\mathcal{M}(M)^*)$.

Corollary 1: $O(K) \cong \Omega^1$.

Pf : send $f \longmapsto f\omega$, check: $(f\omega) = (f) + (\omega) = (f) + K \geq 0$,

$\eta/\omega \longleftarrow \eta$ etc. $\square$

Now let $D = D_+ - D_-$, with $\begin{cases} D_+ \geq 0 \\ D_- > 0 \end{cases} \Longrightarrow$

$$0 \to O(-D_-) \to O(D) \to \bigoplus i^p_* \mathbb{C}^{d_p^+} \to 0 \quad \Longrightarrow$$

$$(G.13) \quad 0 \to \mathcal{L}(D) \to \mathbb{C}^{d^+} \to \ell(-D_-) \to \ell(D) \to 0$$

Now (G.9) $\Rightarrow$

$$\ell(D) - i(D) = d + 1 - g_a \quad \text{for } D \geq 0,$$

where $g_a := \dim H^1(O)$. For the remaining cases of D, (G.13) implies

$$\ell(D) - i(D) = d^+ - i(-D_-) \underset{\overset{\Uparrow}{(G.11)}}{=} d^+ - (d^- + g_a - 1) \underset{\overset{\uparrow}{d=d^+-d^-}}{=} d+1-g_a .$$

<u>Riemann-Roch Theorem</u>: For all $D \in \mathrm{Div}(M)$,

$$\ell(D) - i(D) = d+1-g_a .$$

<u>Remark 1</u>: While (G.12) was proved for $D \geq 0$, it's clear now that it holds in general. (For $i(-D)$, this means showing $\ell(-D) = 0$ for d suff. large; but already for $d > 0$ $\ell(-D)$ is zero since $\deg((f)) = 0$ always.) □

To understand $i(D)$ and g_a, we have to do some heavy lifting. Multiplication induces a map of sheaves

$$\mathcal{O}(-D) \otimes \Omega^1(D) \to \Omega^1$$

hence a pairing

$$\begin{tikzcd} H^1(\mathcal{O}(-D)) \otimes H^0(\Omega^1(D)) \arrow[r] & H^1(\Omega^1) \arrow[r, "\cong", "\text{Dolbeault}"'] \arrow[rr, bend right, "\epsilon"'] & H^{1,1}_{\bar\partial}(M,\mathbb{C}) \arrow[r, "\int_M"] & \mathbb{C} \end{tikzcd} ;$$

one can think of this as a map

$$\begin{aligned} \Theta_D : \Omega^1(D)(M) &\longrightarrow (H^1(\mathcal{O}(-D)))^\vee \\ \omega &\longmapsto \{\xi \mapsto \epsilon(\xi\omega)\} \end{aligned}$$

<u>Theorem 1 (Serre duality for curves)</u>: Θ_D is an isomorphism $(\forall D)$.

Proof: For $E \geq D$ we have

$$0 \to \mathcal{O}(-E) \to \mathcal{O}(-D) \to \oplus\, \iota^{p}_{*}\, \mathbb{C}^{e-d} \to 0$$

$$\Rightarrow \quad \ldots \to H^1(\mathcal{O}(-E)) \twoheadrightarrow H^1(\mathcal{O}(-D)) \to 0$$

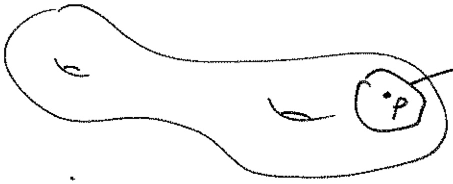
$$\Rightarrow \quad H^1(\mathcal{O}(-D))^\vee \hookrightarrow H^1(\mathcal{O}(-E))^\vee$$

Set $V := \varinjlim_{D} H^1(\mathcal{O}(-D))^\vee$.

$\boxed{\text{Claim 1}}$ $\Theta : \mathcal{K}^1(M) \to V$ is injective.

Pf: Let $\omega \in \Omega^1(D)(M)$

${}_o{}^{*}\, k = -(1+\nu_p(\omega)) \quad (\leq d_p - 1)$



$U_1 \cap |D| = \{p\}$

$U_2 := M \setminus \{p\}$

U_{12} = punctured disk

Introduce a class $\xi \in H^1(\mathcal{O}(-D))$ by taking

$$\xi_{12} := z^k \in \mathcal{O}(-D)(U_{12}) \; ; \quad \text{while } \{\mathfrak{U}_i\}$$

is not a good open cover, this won't affect anything.*

[Note: for z^k to extend to $\mathcal{O}(-D)(U_1)$, trivializing ξ, we'd need $k \geq d_p$.]

Let $\begin{cases} g_2 \in C^\infty(U_2) & \text{extend } \xi \\ g_1 = 0 & \text{on } U_1 \end{cases}$; then $\begin{cases} \check{\delta} g \text{ trivializes } \xi \text{ as an element of } H^1(\mathcal{E}^\infty), \text{ and} \\ \bar{\partial} g \text{ gives the Dolbeault representative of } \xi \text{ in } A^{0,1}(-D). \end{cases}$

and

$$\epsilon(\xi\omega) = \int_M \omega \wedge \bar{\partial} g = \int_{M \setminus U_1} \omega \wedge \bar{\partial} g_2 \quad (\text{b/c } \bar{\partial} g_2 = \bar{\partial} z^k = 0 \text{ on } U_{12})$$

$$= -\int_{M \setminus U_1} d(g_2 \omega) = \int_{\partial U_1} z^k \omega = 2\pi i \, \mathrm{Res}_p(z^k \omega) \neq 0 \text{ by choice of } k.$$

So ω defines a nonzero element of V. //

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* in particular, recall that $\check{H}^1(\mathcal{O}(-D))$ is defined as $\varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, \mathcal{O}(-D))$, so one can use any $\mathfrak{U}$ to define a class. By ξ we mean its image in the limit.

$\boxed{\text{Claim 2}}$ $\omega \in K^1(M)$ maps into $H^1(\mathcal{O}(-D))^\vee \iff \omega \in \Omega^1(D)(M)$.

Pf: If $\theta(\omega) \in H^1(\mathcal{O}(-D))^\vee$, then it has to vanish on all coboundaries for this group; if also $-(k+1) = v_p(\omega) < -d_p$, then $k \geq d_p \Rightarrow$ $\left\{ \begin{array}{l} [\xi] = 0 \text{ in above, but} \\ \text{pairing } \epsilon(\xi\omega) \neq 0 \end{array} \right.$ ⨳. So $v_p(\omega) \geq -d_p$ $(\forall p)$. //

It remains to prove θ_D surjective. First note

- $K^1(M) \cong$ 1-dim'l $\mathcal{M}(M)$-vector space
- $V = \mathcal{M}(M)$-vector space (given $v(\cdot) \in V$, $(f.v)(\xi) := v(f\xi)$)
- θ is $\mathcal{M}(M)$-linear $\left(\theta(f\omega)(\xi) := \epsilon(\omega f \xi) = \theta(\omega)(f\xi) = (f.\theta(\omega))(\xi) \right)$

Now given $\phi \in H^1(\mathcal{O}(-D))^\vee$ $(\in V)$, it will suffice to prove the

$\boxed{\text{Claim 3}}$ $\phi = \theta(\tilde{\omega})$ for some $\tilde{\omega} \in K^1(M)$.

Pf.: by (G.12)(+ Remark 1), for $n \gg 0$

$$\dim H^1(\mathcal{O}(-D-n[p]))^\vee = n + O(1).$$

Moreover, $H^1(\mathcal{O}(-D-n[p]))^\vee$ contains

- $\theta(\Omega^1(D+n[p])(M))$ $\rightsquigarrow$ $\dim \geq n + O(1)$ by (G.12) & $\boxed{\text{Claim 1}}$

and

- $\mathcal{O}(n[p])(M).\phi$ $\rightarrow$ $\dim \geq n + O(1)$ by (G.12)

So for n suff. large, these subspaces intersect nontrivially:

$$\exists \left\{ \begin{array}{l} f \in \mathcal{O}(n[p])(M) \\ \omega \in \Omega^1(D+n[p])(M) \end{array} \right. \text{ s.t. } f.\phi = \theta(\omega)$$

$\Longrightarrow$ (act by f^{-1}) $\phi = f.\theta(\omega) = \theta(\underbrace{f\omega}_{\text{our } \tilde{\omega}})$, done. //

By $\boxed{\text{Claim 2}}$, $\tilde{\omega}$ must automatically lie in $\Omega^1(D)(M)$. □

Here come the Corollaries !

Corollary 2: (i) $i(D) = \dim\left(\Omega^1(-D)(M)\right)$

(ii) $H^0(\mathcal{O}(D)) \cong H^1(\Omega^1(-D))$

(iii) $\epsilon : H^1(\Omega^1) \to \mathbb{C}$ is an isomorphism

Pf.: (i) is immediate from Thm. 1

(ii) is $H^0(\Omega^1(D-K)) \cong H^0(\mathcal{O}(K-D))$

(iii) is the special case $D = 0$ of (ii). $\square$

Now we have $H^0(\Omega^1)^\vee \cong H^1(\mathcal{O}) \Rightarrow \dim \Omega^1(M) = g_a$. Moreover,

$$0 \to \mathbb{C} \to \mathcal{O} \to \Omega^1 \to 0$$

gives

$$\begin{tikzcd}
 & & H^0(\mathcal{O}) \arrow[d, equal] & & & & & & \mathbb{C} \arrow[d, phantom, "\cong" sloped] & \mathbb{C} \arrow[d, equal] & \\
0 \arrow[r] & \mathbb{C} \arrow[r, "\cong"'] & \mathbb{C} \arrow[rr] \arrow[dr] & & \Omega^1(M) \arrow[r] & H^1(\mathbb{C}) \arrow[r] & H^1(\mathcal{O}) \arrow[rr] \arrow[dr] & & H^1(\Omega^1) \arrow[r, "\cong"] & H^2(\mathbb{C}) \arrow[r] & 0 \\
 & & & 0 \arrow[ur] & & & & 0 \arrow[ur] & & &
\end{tikzcd}$$

$\Rightarrow 2g_a = 2g \Rightarrow g_a = g$.

Corollary 3: (i) $\dim \Omega^1(M) = g$

(ii) [R-R] $\ell(D) - i(D) = d - g + 1$

(iii) $H^2(M, \mathbb{C}) \cong \underset{\text{"}H^{1,1}\text{"}}{H^1(\Omega^1)}$

Remark 2: Given a finite set of principal parts $\left\{ \frac{a_{-1}^{(i)}}{z_{p_i}} + \cdots + \frac{a_{-d_i}^{(i)}}{(z_{p_i})^{d_i}} \right\}$ on M, can we solve the Mittag-Leffler problem for $K^1(M)$? Well, it's clear from the Residue theorem that a necessary condition is

$$(*) \quad \boxed{\sum_i a_{-1}^{(i)} = 0}.$$

Let $D = \sum_i d_i[p_i]\ (\geq 0)$; then $\ell(-D) = 0$, so by R-R $\dim H^0(\Omega^1(D)) = g + \sum_i d_i - 1$, and the dimensions of the subspace of principal parts spanned by these forms is

$$\dim H^0(\Omega^1(D)) - \dim H^0(\Omega^1) = \sum d_i - 1.$$

That means $(*)$ is also a <u>sufficient</u> condition! $\square$

Now we have a map

$$(6.14) \qquad \Omega^1(M) \oplus \overline{\Omega^1(M)} \longrightarrow H^1_{dR}(M,\mathbb{C})$$

induced by $\qquad (\omega, \bar\varphi) \longmapsto [\omega + \bar\varphi]$,

as well as a (compatible) map $\overline{\Omega^1}(M) \to \dfrac{A^{0,1}(M)}{\bar\partial A^0(M)}$.

$$\left(\begin{tikzcd} \overline{\Omega^1}(M) \arrow[r] & \dfrac{A^{0,1}(M)}{\bar\partial A^0(M)} \\ \Omega^1(M)\oplus\overline{\Omega^1(M)} \arrow[r] \arrow[u, two heads] & \dfrac{A^1(M)_{d\text{-closed}}}{dA^0(M)} \arrow[u, "\text{take } (0,1) \text{ part}"'] \end{tikzcd}\right)$$

<u>Corollary 4</u>: (i) [Hodge decomposition] $H^1_{dR}(M,\mathbb{C}) \cong \underset{\text{"}H^{1,0}\text{"}}{\Omega^1(M)} \oplus \underset{\text{"}H^{0,1}\text{"}}{\overline{\Omega^1(M)}}$

(ii) $\overline{\Omega^1(M)} \cong H^1(\mathcal{O}) \left(\underset{\text{Serre}}{\cong} \Omega^1(M)^\vee\right)$

<u>Proof</u>: (i) We need to check (6.14) <u>injective</u> ($\cong$ then follows from equality of dims.). Suppose $\omega + \bar\varphi = df$, $f \in C^\infty(M)$. Then

$$0 \underset{\text{(stokes)}}{=} -\int \underbrace{d(f\varphi)}_{df\wedge\varphi} = \int_M \varphi \wedge \bar\varphi$$

$$\Longrightarrow \varphi \equiv 0 \Longrightarrow \omega = df \Longrightarrow \frac{\partial f}{\partial \bar z} = 0 \Longrightarrow f \in \mathcal{O}(M)$$

$$\underset{\text{MMP}}{\Longrightarrow} \omega = 0.$$

(ii) We have a diagram (exact rows)

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1(M) \arrow[r] & H^1(\mathbb{C}) \arrow[r] & H^1(\mathcal{O}) \arrow[r] & 0 \\ 0 \arrow[r] & \Omega^1(M) \arrow[r] \arrow[u, equal] & \Omega^1(M)\oplus\overline{\Omega^1(M)} \arrow[r] \arrow[u, "\wr"'] & \overline{\Omega^1(M)} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

$\xRightarrow{\text{5-lemma}}$ last arrow is an $\cong$. $\square$

Ex/ **Corollary 5**: (a) $d > 2g-2 \Longrightarrow i(D) = 0$

(b) $d < 0 \Longrightarrow \ell(D) = 0$ //

Let $D = k[p] - \sum_{i=1}^{\ell} [p_i]$, $k - \ell > 2g-2$. Then R-R + Cor. 5 $\Longrightarrow$

(G.15) $$\ell(D) = k - \ell - g + 1 .$$

Corollary 6: $\exists$ a holo. embedding $\varphi : M \hookrightarrow \mathbb{P}^{g+1}$.

Proof: Fix $p \in M$. Then for arbitrary* $q, r \in M$,

(G.16) $$\underset{\mathcal{L}_{q,r}}{\underset{\shortparallel}{\mathcal{L}((2g+1)[p]-[q]-[r])}} \subsetneq \underset{\mathcal{L}_{q}}{\underset{\shortparallel}{\mathcal{L}((2g+1)[p]-[q])}} \subsetneq \underset{\mathcal{L}}{\underset{\shortparallel}{\mathcal{L}((2g+1)[p])}}$$

	$\mathcal{L}_{q,r}$	$\mathcal{L}_q$	$\mathcal{L}$
degree of divisor =	$\boxed{2g-1}$	$\boxed{2g}$	$\boxed{2g+1}$
dimension of space = (G.15)	g	$g+1$	$g+2$

Let $\{f_0, \dots, f_{g+1}\} \subset \mathcal{L}$ be a basis, and define for $m \in M \setminus \{p\}$

(G.17) $$\varphi(m) := [f_0(m) : \cdots : f_{g+1}(m)] \in \mathbb{P}^{g+1}.$$

For $m = p$ this is unsuitable, since (except for constants) the functions will blow up. Writing z for a local holo. coord. at p, set

(G.18) $$\varphi(p) := [(z^{2g+1} f_0)(p) : \cdots : (z^{2g+1} f_{g+1})(p)] \in \mathbb{P}^{g+1} .$$

We already have $\lim_{m \to p} \varphi(m) = \varphi(p)$ and have constructed an analytic map between complex manifolds M and $\mathbb{P}^{g+1}$, provided (G.17-18) do not yield $[0 : \cdots : 0]$ at any point of M.

To check this doesn't happen, and that φ is injective,

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* they could be equal, or equal to p.

we use (6.16) :

- for $q \neq p$, $\mathcal{L}_q \subsetneq \mathcal{L}$ $\Rightarrow$ $\exists\, f \in \mathcal{L}$ not vanishing at q
  $\Rightarrow$ not all $f_i(q) = 0$
- $\mathcal{L}_p \subsetneq \mathcal{L}$ $\Rightarrow$ $\exists\, f \in \mathcal{L}$ with $\nu_p(f) = -(2g+1)$ $\Rightarrow$ not all $(z^{2g+1} f_i)(p) = 0$
- for p, q, r distinct $\mathcal{L}_{q,r} \subsetneq \mathcal{L}_q$ $\Rightarrow$ $\exists\, f \in \mathcal{L}$ vanishing at q but not at r
  $\Rightarrow$ $\varphi(q) \neq \varphi(r)$
- for $q \neq p$, $\mathcal{L}_{p,q} \subsetneq \mathcal{L}_q$ $\Rightarrow$ $\exists\, f \in \mathcal{L}$ vanishing at q but with $(z^{2g+1} f)$ <u>not</u> vanishing at p $\Rightarrow$ $\varphi(q) \neq \varphi(p)$. □

<u>Remark 3</u>: (i) This can be refined further to show $M \hookrightarrow \mathbb{P}^3$ and $M \underset{\text{(immerses)}}{\looparrowright} \mathbb{P}^2$ with only "normal crossing" or "ODP" singularities (locally of form $xy = 0$)

(ii) One can also use a basis for $\Omega^1(M)$ to get an embedding in $\mathbb{P}^{g-1}$ ("canonical curve"), but this doesn't work for <u>hyperelliptic RS</u>'s — those with a degree-2 map to $\mathbb{P}^1$.

(iii) In fact, we can even check that the image of the above mapping $M \to \mathbb{P}^{g+1}$ will be smooth :

$\mathcal{L}_{q,q} \subsetneq \mathcal{L}_q$ $\Rightarrow$ $\exists\, f \in \mathcal{L}_q$ vanishing to exactly 1st order at q
$\Rightarrow$ derivative of φ (in local coords.) is nonzero there
+ similar check at p. □

Before turning to periods and the Riemann bilinear relations, here is one more important application of sheaf technology. Let

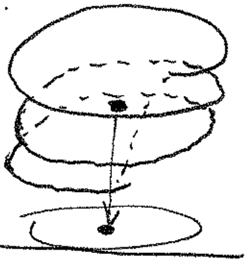
$$\pi : M \to N$$

be a surjective mapping of CRS's (most commonly N will be $\mathbb{P}^1$) of degree d_π (:= cardinality of $\pi^{-1}(q)$ for general N). Let

$\underline{\Delta} = \{p_1, \dots, p_m\} \subset M$ denote the set of ramification points, of degrees r_i (i.e. locally the map looks like $z \mapsto z^{r_i}$); we have $\pi(\underline{\Delta}) = \{q \in N \,|\, |\pi^{-1}(q)| < d_\pi\}$. Define the <u>ramification divisor</u>

$$\begin{cases} R_\pi := \sum_i (r_i - 1)[p_i] \in \mathrm{Div}(M) \\ r_\pi := \deg R_\pi \end{cases}.$$

<u>Picture of $r_i = 3$</u>:



"Cyclic parking-lot"

Consider $\omega \in K^1(N) \setminus \{0\}$, with $(\omega) \cap \pi(\underline{\Delta}) = \emptyset$. Then under $z \mapsto z^{r_i} = w$, dw pulls back to $r_i z^{r_i - 1} dz$, and so we have for $\pi^* \omega \in K^1(M)$

the obvious "preimage divisor"

$$(\pi^* \omega) = \boxed{\pi^*(\omega)} + R_\pi$$

$$\Rightarrow 2g_M - 2 \underset{\text{P-H}}{=} \deg((\pi^*\omega)) = \deg(\pi^*(\omega)) + \deg R_\pi$$

$$= d_\pi \cdot \underbrace{\deg((\omega))}_{2g_N - 2} + r_\pi,$$ giving the

<u>Riemann - Hurwitz formula</u>: $\boxed{g_M = d_\pi \cdot (g_N - 1) + \frac{r_\pi}{2} + 1.}$

<u>Example 2</u>: For a hyperelliptic RS $M \xrightarrow[\text{off } \underline{\Delta}]{2:1} \mathbb{P}^1$, with $2b$ branch pts., $g_M = b - 1$. //

Ex/ Always $g_M \geq g_N$.

Note the restrictions imposed by this formula on the possible data involved: we <u>must</u> have

$$\begin{cases} 2 \,|\, r_\pi \\ d_\pi \,|\, g_M - \frac{r_\pi}{2} - 1 \end{cases}$$

Periods

Let $\{\omega_1, \dots, \omega_g\} \subset \Omega^1(M)$ be a basis

$\{\gamma_1, \dots, \gamma_{2g}\} \subset H_1(M, \mathbb{Z})$ the symplectic basis described above

Then we have the period vectors $\pi_j := \begin{pmatrix} \int_{\gamma_j}\omega_1 \\ \vdots \\ \int_{\gamma_j}\omega_g \end{pmatrix} \in \mathbb{C}^g$ and the period matrix $\Pi := \begin{pmatrix} \uparrow & & \uparrow \\ \pi_1 & \cdots & \pi_{2g} \\ \downarrow & & \downarrow \end{pmatrix}$.

Proposition 2: The $\{\pi_j\}_{j=1}^{2g}$ are $\mathbb{R}$-linearly independent (viewed as vectors in $\mathbb{R}^{2g}$).

Proof: If $\underline{0} = \Pi\,\underline{a}$ $(\underline{a} \in \mathbb{R}^{2g}\setminus\{0\})$ then $\underline{0} = \underbrace{\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}}_{2g\times 2g} \underline{a} \Longrightarrow$

$$\operatorname{rank}\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} < 2g \Longrightarrow \exists\, \underline{b} \in \mathbb{C}^{2g}\setminus\{\underline{0}\} \text{ s.t. } {}^t\underline{b}\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = {}^t\underline{0} \Longrightarrow$$

$$(\forall j)\ \int_{\gamma_j}\Big(\underbrace{\sum_{i=1}^{g} b_i\omega_i}_{=:\omega} + \underbrace{\sum_{i=1}^{g} b_{g+i}\overline{\omega_i}}_{=:\overline{\varphi}}\Big) = 0 \xRightarrow[\text{Theorem}]{dR} [\omega+\overline{\varphi}] = 0 \text{ in } H^1_{dR} \xRightarrow[\text{Cor. 4}]{}$$

$$\omega, \overline{\varphi} = 0 \Longrightarrow \underline{b} = 0 \text{ ⨳}.$$

□

So $\Lambda_M := \mathbb{Z}\langle \pi_1, \dots, \pi_{2g}\rangle \subset \mathbb{C}^g$ is a full lattice, or more intrinsically $H_1(M, \mathbb{Z}) \subset \Omega^1(M)^\vee$.

Definition 2: The Jacobian of M is the complex g-torus

$$J(M) := \frac{\Omega^1(M)^\vee}{H_1(M,\mathbb{Z})} \underset{\substack{\uparrow \\ \text{evaluate against} \\ \text{the basis } \{\omega_1,\dots,\omega_g\}}}{\xrightarrow{\cong}} \frac{\mathbb{C}^g}{\Lambda_M}$$

□

Now for any $\psi \in \Omega^1(M) \oplus \overline{\Omega^1(M)}$, we have the equality of functionals on H_1

$$(G.19) \qquad [\psi] = \sum_{j=1}^{g} \left(\pi_j(\psi)\,[\gamma_{j+g}] - \pi_{j+g}(\psi)\,[\gamma_j] \right)$$

where we are using the equivalent pairings $\langle\, ,\, \rangle$ from (G.4). For $\omega, \varphi \in \Omega^1(M)$

$$(G.20) \qquad 0 = \int_M \overset{\substack{\text{type }(2,0) \Rightarrow \text{zero} \\ (\dim M = 1)}}{\omega \wedge \varphi} = \langle [\omega], [\varphi] \rangle \underset{(G.19)}{=\!=} \sum_{j=1}^{g} \left(\pi_j(\varphi)\,\pi_{j+g}(\omega) - \pi_{j+g}(\varphi)\,\pi_j(\omega) \right)$$

$$(G.21) \qquad 0 < \underbrace{i \int_M \omega \wedge \bar\omega}_{\substack{\text{~~locally~~},\ i\,dz \wedge d\bar z = \\ 2\,dx \wedge dy \text{ (positively} \\ \text{oriented)}}} = i \langle [\omega], \overline{[\omega]} \rangle \overset{+ (G.1)}{=\!=} -i \sum_{j=1}^{g} \left(\overline{\pi_j(\omega)}\,\pi_{j+g}(\omega) - \overline{\pi_{j+g}(\omega)}\,\pi_j(\omega) \right)$$

Substituting $\begin{cases} \omega = \omega_\ell \\ \varphi = \omega_k \end{cases}$ into (G.20-21), we can recast the result in matrix form.*

$$(G.22) \qquad \begin{cases} (i) & \Pi\, J\, {}^t\Pi = 0 \\ (ii) & \sqrt{-1}\, \Pi\, J\, {}^t\overline{\Pi} > 0 \end{cases}$$

which is invariant under symplectic change of γ-basis.

Writing $\Pi = (\underset{g \times g}{A} \quad \underset{g \times g}{B})$, (G.22) becomes

$$(G.23) \qquad \begin{cases} (i) & A^t B - B^t A = 0 \\ (ii) & \sqrt{-1}\,(A^t \bar B - B^t \bar A) > 0 \end{cases}$$

in particular $\sqrt{-1}\ {}^t\underline{v}\,(A^t \bar B - B^t \bar A)\,\underline{\bar v} > 0 \quad \forall\, \underline{v} \neq \underline{0} \implies$ A cannot have nontrivial left-kernel $\implies$ A is invertible.

---

* e.g.,
$$0 = \Pi\, J\, {}^t\Pi = \begin{pmatrix} \uparrow & & \uparrow \\ \pi_1 & \cdots & \pi_{2g} \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} \begin{pmatrix} \leftarrow \pi_1 \rightarrow \\ \\ \leftarrow \pi_{2g} \rightarrow \end{pmatrix} = \underbrace{\begin{pmatrix} \uparrow & & \uparrow & \uparrow & & \uparrow \\ -\pi_{g+1} & \cdots & -\pi_{2g} & \pi_1 & \cdots & \pi_g \\ \downarrow & & \downarrow & \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \leftarrow \pi_1 \rightarrow \\ \vdots \\ \leftarrow \pi_{2g} \rightarrow \end{pmatrix}}_{\substack{\text{entries of the product corr. to} \\ (G.20) \text{ with the substitutions } \omega_\ell,\ \omega_k}}$$

Using A^{-1} to change our ω-basis,* we have

$$\Pi' = A^{-1}\Pi = \Big(I_g \quad \underbrace{A^{-1}B}_{=:\,Z} \Big) \quad \Longrightarrow$$

(G.23) with $A = I_g$, $B = Z$

$$\begin{cases} \text{(i)} & {}^tZ - Z = 0 \\ \text{(ii)} & \sqrt{-1}\,\underbrace{({}^t\bar{Z} - Z)}_{\bar{Z} - Z} > 0 \end{cases}$$

and hence the

Theorem 2:

$$\boxed{\text{Riemann Bilinear Relations } \text{~~(for curves)~~}: \quad \text{(I)}\ Z = {}^tZ \qquad \text{(II)}\ \mathrm{Im}(Z) > 0}$$

* * *

Complex tori

Now we shall approach these "bilinear relations" from within a different (but related) context. Let $\Lambda \subset V \cong \mathbb{C}^n$ be a full lattice, $T := V/\Lambda$ the complex n-torus. Clearly $V \cong T_{T,\underline{0}}$, and we also write $W := V^{\mathbb{R}}$. There are 2 natural choices of basis for $W_{\mathbb{C}}^{\vee} \left(= T_{T,0}^{\vee} \otimes_{\mathbb{R}} \mathbb{C}\right)$:

- given a basis $\lambda_1, \ldots, \lambda_{2n}$ of Λ (hence of W & $W_{\mathbb{C}}$), let $\gamma^{\vee} := \{dx_1, \ldots, dx_{2n}\}$ denote the dual basis of $W_{\mathbb{C}}^{\vee}$.

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* viz., $\Pi = \underbrace{\begin{pmatrix}\omega_1 \\ \vdots \\ \omega_g\end{pmatrix}(\gamma_1 \cdots \gamma_{2g})}_{\text{where "product" of } \omega_i \,\&\, \gamma_j \text{ is just } \int_{\gamma_j}\omega_i} \Longrightarrow A^{-1}\Pi = \underbrace{A^{-1}\begin{pmatrix}\omega_1 \\ \vdots \\ \omega_g\end{pmatrix}}_{=:\begin{pmatrix}\omega_1' \\ \vdots \\ \omega_g'\end{pmatrix}}(\gamma_1 \cdots \gamma_{2g})$

- $W_{\mathbb{C}} \cong W_+ \oplus W_-$ ($\cong V \oplus \bar{V}$, where $V \xrightarrow{\cong} W_+$ is $v \mapsto v - i\underset{\uparrow \text{cx. str.}}{I}(v)$)

$$\Downarrow$$

$$W_{\mathbb{C}}^\vee \cong W_+^\vee \oplus W_-^\vee \cong \underbrace{\mathbb{C}\langle \overbrace{dz_1, \dots, dz_n}^{\text{some basis for } W_+} \rangle \oplus \mathbb{C}\langle d\bar{z}_1, \dots, d\bar{z}_n \rangle}_{\overset{\shortparallel}{\mathfrak{z}}}$$

One can obviously extend these to translation-invariant differential forms on T, and this leads to isomorphisms (as in problem set 1 Ex. 6 solution)

$$\Lambda^k W_{\mathbb{C}}^\vee \xrightarrow{\cong} H^k_{dR}(T, \mathbb{C})$$

for each k. In fact, using the (p,q)-decomposition of the left-hand side ($\cong \bigoplus_{p+q=k} \Lambda^p W_+^\vee \otimes \Lambda^q W_-^\vee$) we get the

<u>Proposition 3</u>: (Hodge decomposition)

$$H^k_{dR}(T, \mathbb{C}) \cong \bigoplus_{p+q=k} \underbrace{\mathbb{C}\left\langle \{dz_I \wedge d\bar{z}_J\}_{\substack{|I|=p \\ |J|=q}} \right\rangle}_{\text{"}H^{p,q}(T)\text{"}}$$

//

It is also clear that we have

$$H^k(T, \mathbb{Z}) \cong \mathbb{Z}\langle \{dx_K\}_{|K|=k} \rangle.$$

Recall that a necessary* condition for T to possess a

---

* and, taking for granted a form of the Kodaira embedding theorem, sufficient

projective embedding, is for it to have a Kähler metric whose Kähler class is integral — or equivalently, a d-closed (wolog translation-invariant) positive (1,1)-form with integral class in H^2. This is called a polarizing form, and we look for conditions (essentially on Λ) under which one exists.

Start by identifying

$$H_1(T,\mathbb{Z}) \xleftarrow{\cong} \Lambda$$
$$[\gamma_j] := \longleftarrow \lambda_j \quad ,$$

where we think of γ_j as a straight segment in V from 0 to $\lambda_j \in \Lambda$, and dz_i as differentials of coordinates on $V \cong \mathbb{C}^n$. In fact, with these identifications we have

$$T \left(= \frac{V}{\Lambda}\right) \cong \boxed{\frac{\Omega^1(T)^\vee}{H_1(T,\mathbb{Z})}}$$

Define the $(n\times 2n)$ period matrix

$$\Pi = \left\{ \pi_{ij} = \int_{\gamma_j} dz_i \right\},$$

so that the change-of-basis matrix

$${}_{\gamma^\vee}[\mathrm{id}]_{\eta} = \left({}^t\Pi \;\; {}^t\overline{\Pi} \right) =: \tilde{\Pi}$$

(identity on $W_{\mathbb{C}}^\vee = H^1(T,\mathbb{C})$)

this means simply that $\begin{cases} dz_i = \sum_j \pi_{ij}\, dx_j \\ d\bar{z}_i = \sum_j \overline{\pi_{ij}}\, dx_j \end{cases}$

with inverse

$${}_{\eta}[\mathrm{id}]_{\gamma^\vee} = \tilde{\Pi}^{-1} = \begin{pmatrix} \Xi \\ \overline{\Xi} \end{pmatrix}$$

$n\times 2n$, with entries $\{\xi_{ij}\}$

(so that

$dx_j = \sum \xi_{ij} dz_j + \sum \overline{\xi_{ij}} d\bar{z}_j$ or, writing $\underline{dx} = \begin{pmatrix} dx_1 \\ \vdots \\ dx_{2n} \end{pmatrix}$ and

$\widetilde{\underline{dz}} = \begin{pmatrix} dz_1 \\ \vdots \\ dz_n \\ d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{pmatrix} = \begin{pmatrix} \underline{dz} \\ \underline{d\bar{z}} \end{pmatrix}$, ${}^t\widetilde{\Pi}^{-1}\widetilde{\underline{dz}} = \underline{dx}$) .

Let ω be a translation-invariant, d-closed 2-form with integral cohomology class: so

$$\omega = \frac{1}{2} \sum q_{ij}\, dx_i \wedge dx_j \qquad (\{q_{ij}\} = Q \quad 2n \times 2n \text{ skew-symmetric matrix}).$$

We will quote the linear algebra

<u>Lemma 1</u>: We may choose the (integral) $\lambda/\gamma/\gamma^\vee/dx$ - basis so that

$$Q = Q_\delta := \begin{pmatrix} 0 & \Delta_\delta \\ -\Delta_\delta & 0 \end{pmatrix}, \text{ where}$$

$\Delta_\delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}$ and $\delta_1 | \delta_2 | \cdots | \delta_n$ $(\in \mathbb{N})$ are invariants of Q. //

If $\omega > 0$ then (by Corollary E.2) $\omega^n \neq 0 \Rightarrow \underbrace{\text{all } \delta_i \neq 0}$.

Assume this henceforth.

Now then

$$\omega = \frac{1}{2}\, {}^t\underline{dx}\, Q\, \underline{dx} = \frac{1}{2}\, {}^t\widetilde{\underline{dz}}\, \widetilde{\Pi}^{-1} Q\, {}^t\widetilde{\Pi}^{-1} \widetilde{\underline{dz}}$$

$$= \frac{1}{2} \left({}^t\underline{dz} \;\; {}^t\underline{d\bar{z}} \right) \left(\begin{array}{c|c} \Xi Q\, {}^t\Xi & \boxed{\Xi Q\, {}^t\overline{\Xi}} \\ \hline \overline{\Xi} Q\, {}^t\Xi & \overline{\Xi} Q\, {}^t\overline{\Xi} \end{array} \right) \begin{pmatrix} \underline{dz} \\ \underline{d\bar{z}} \end{pmatrix}$$

Clearly, calling this $=: i\mathcal{H}$,

${}^tQ = -Q \Rightarrow$

${}^t\mathcal{H} = \overline{\mathcal{H}}$

and ω is $\left\{\begin{array}{l}\text{of type } (1,1)\\ \text{and } > 0\end{array}\right.$ $\underset{\substack{\text{pf of}\\ \text{Theorem E.1}}}{\Longleftrightarrow}$ $\tilde{\Pi}^{-1} Q\, {}^t\tilde{\Pi}^{-1} = i\begin{pmatrix} 0 & \mathcal{H} \\ -\overline{\mathcal{H}} & 0 \end{pmatrix}$

with $\mathcal{H} > 0$

$\underset{\substack{\text{take inverse}\\ H = \mathcal{H}^{-1}\\ (\check{0} \Leftrightarrow \check{0})}}{\Longleftrightarrow}$ ${}^t\tilde{\Pi}\, Q^{-1}\, \tilde{\Pi} = -i\begin{pmatrix} 0 & -\overline{H} \\ H & 0 \end{pmatrix}$

with $H > 0$.

Writing $\Pi = (A \;\; B)$ as in the CRS setting, we get

$$i\,{}^t\tilde{\Pi}\, Q^{-1}\, \tilde{\Pi} = i \begin{pmatrix} A & B \\ \bar{A} & \bar{B} \end{pmatrix} \begin{pmatrix} 0 & -\Delta_{\delta^{-1}} \\ \Delta_{\delta^{-1}} & 0 \end{pmatrix} \begin{pmatrix} {}^tA & {}^t\bar{A} \\ {}^tB & {}^t\bar{B} \end{pmatrix}$$

$$= i \left(\begin{array}{c|c} B\Delta_{\delta^{-1}}{}^tA - A\Delta_{\delta^{-1}}{}^tB & * \\ \hline \bar{B}\Delta_{\delta^{-1}}{}^tA - \bar{A}\Delta_{\delta^{-1}}{}^tB & * \end{array}\right)$$

so that $H > 0 \Longleftrightarrow i\left(\bar{B}\,\Delta_{\delta^{-1}}{}^tA - \bar{A}\,\Delta_{\delta^{-1}}{}^tB\right) > 0 \Longrightarrow \underbrace{A \text{ invertible}}_{\text{Assume this henceforth}}.$

$\Longrightarrow$ wolog wma $\Pi = (\Delta_\delta \;\; Z)$.

Hence, our 2 conditions on ω become $Z\Delta_{\delta^{-1}}{}^t\Delta_\delta - \Delta_\delta\Delta_{\delta^{-1}}{}^tZ = 0$ $\Rightarrow Z = {}^tZ$, and $i(\bar{Z} - Z) > 0$. We conclude

Theorem 3: T admits a polarizing form $\Longleftrightarrow$

$\exists$ bases for $H_1(T, \mathbb{Z})\ (\cong \Lambda)$ and $\Omega^1(T)\ (\cong V^\vee)$ such that the period matrix Π is of the form $(\Delta_\delta \;\; Z)$ with

$$\boxed{\operatorname{Im} Z > 0 \quad \text{AND} \quad {}^tZ = Z}.$$

<u>Remark 4</u>: (i) These are the conditions under which T admits a projective embedding, i.e. is an abelian variety (and usually denoted A). Clearly, <u>the general one is not</u>.

(ii) There are 2 ways to embed it in $\mathbb{P}^N$ in this case: Kodaira's theorem, or explicit construction of <u>theta functions</u>.

If all $\delta_i = 1$, then ω is called a <u>principal polarization</u> of T.

Given a CRS M, that its Jacobian $J(M) \triangleq \frac{\Omega^1(M)^\vee}{H_1(M,\mathbb{Z})} \left(= \frac{V}{\Lambda}\right)$ is polarized is a tautology; in fact, by taking $Q = \underset{\left(\substack{\text{intersection}\\ \text{form on } M}\right)}{J}$ it is principally polarized.

Moduli

The g^{th} Siegel upper half-space*

$$\mathfrak{h}_g := \left\{ Z \in M_{g\times g}(\mathbb{C}) \;\middle|\; {}^tZ = Z \text{ and } \mathrm{Im}(Z) > 0 \right\}$$

parametrizes the set of <u>principally polarized abelian varieties</u> (PPAV's).

Over $\overset{(\text{matrix})}{\tau} \in \mathfrak{h}_g$, we have $A_\tau = \mathbb{C}^g / \Lambda_\tau$ where $\Lambda_\tau := \mathbb{Z}\left\langle \substack{\text{columns of}\\ (I \mid Z)} \right\rangle$.

In order that each isomorphism class of PPAV occur only once, we take the quotient by $\Gamma_g := Sp_{2g}(\mathbb{Z})$

---

* $\mathfrak{h}_1$ is just the familiar upper half plane. For $g=1$, any τ (s.t. $\mathbb{Z}\langle 1, \tau\rangle$ is a lattice) is in $\pm\mathfrak{h}_1$, so all complex 1-tori are algebraic. (False for $g \geq 2$).

$$\mathcal{A}_g := \Gamma_g \backslash \mathfrak{h}_g$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ acts by $\gamma\langle\tau\rangle := (A\tau+B)(C\tau+D)^{-1}$. This action on the "period point" is equivalent to a symplectic-linear change of integral basis; this preserves the polarizing form.

Remark 5: The genuine miracle here is that $\mathcal{A}_g$ has a projective embedding and is therefore an algebraic variety, called the g^{th} Siegel modular variety. The embedding is given by Siegel modular forms (more on these later). //

What about moduli of curves? First some more general discussion.

Kodaira & Spencer developed the theory of deformations of complex structures on manifolds. Since manifolds are made up of charts, a "deformation" should be a shifting of the chart (on the overlaps). The "tangent to a deformation" should therefore identify with a class in $H^1(M, \Theta^1_M)$, where $\Theta^1_M := \mathcal{O}(T^{(1,0)}_M)$ is the sheaf of holomorphic vector fields.

To be more precise, suppose $\pi : \mathcal{X} \to \mathbb{D}$ is a submersive holomorphic map from a complex n-manifold onto the unit disk, with compact fibers. Then the $\{X_t := \pi^{-1}(t)\}_{t\in\mathbb{D}}$ are a family of compact complex $(n-1)$-manifolds, all diffeomorphic* but with varying complex structure. Considering the short-exact sequence of sheaves on $M := X_0$

$$0 \to \Theta^1_M \to \Theta^1_{\mathcal{X}}|_M \to \underbrace{(\pi^* T^{(1,0)}_{D,\{0\}}) \otimes \mathcal{O}_M}_{\cong \mathcal{O}_M} \to 0,$$

the image of $\partial/\partial t \otimes 1$ under the connecting homomorphism

$$H^0(M, (\pi^* T^{1,0}_{D,\{0\}}) \otimes \mathcal{O}_M) \xrightarrow{\delta} H^1(M, \Theta^1_M)$$

defines the Kodaira-Spencer class. This will be used later in the course.

Now consider the case where M is a CRS of genus g. I ask you to believe, for the next proof, that $H^1(M, \Theta^1_M)$ is the "tangent space to $\mathcal{M}_g$ (= moduli space of genus g CRS's) at M". (For a proof of this ~~theorem~~ without such an assumption, see p. 292 of my Alg. Geom. book online.)

Theorem 4 (Riemann): Compact surfaces of genus $g \geq 2$ have (up to isomorphism) $3g-3$ moduli. So

$$\dim(\mathcal{M}_g) = \begin{cases} 3g-3 & g \geq 2 \\ 1 & g = 1 \\ 0 & g = 0 \end{cases}.$$

---

* This is a consequence of Frobenius's theorem. Any C^∞ lifting of the vector field $\partial/\partial t$ on D to $\mathcal{X}$ defines a rank-one, hence integrable, distribution and thus an isomorphism $\mathcal{X} \cong M \times D$ of C^∞ manifolds (it does not respect the complex structures!).

Proof: $H^1(\Theta^1_M) = H^1((\Omega^1_M)^\vee) = H^1(\mathcal{O}(-\overset{\text{(canonical divisor)}}{K})) \underset{\text{Serre}}{\cong} H^0(\mathcal{O}(2K))^\vee$.

The dimension of this space is

for $g \geq 2$: $i(-K) \underset{\uparrow}{=} g - \deg(-K) - 1 = 3g-3$

$\left(\begin{array}{c}\deg(-K) = 2-2g < 0 \text{ for } g \geq 2 \\ \Rightarrow l(-K) = 0\end{array}\right)$

for $g = 1$: $\underbrace{K \cong \mathcal{O}}$ $\Rightarrow$ dim $= 1$

(on $\mathbb{C}/\Lambda$, Ω^1 is trivialized by "dz")

for $g = 0$: $\deg K < 0 \Rightarrow$ dim $= 0$. □

Corollary 7: A general CRS of genus $g \geq 3$ is non-hyperelliptic.

Proof: Hyperelliptic CRS's are determined by the $2g+2$ branch points of their 2:1 mapping to $\mathbb{P}^1$, modulo the action of $PGL_2(\mathbb{C})$ which can send any 3 of these branch pts to $0, 1, \infty$. So they have $2g-1$ moduli, which is $< 3g-3$ for $g \geq 3$. □

Corollary 8: For $g \geq 4$, the general g-dimensional PPAV is not the Jacobian of a CRS.

Proof: First notice that

$$\dim(\mathcal{A}_g) = \dim(\text{space of symmetric } g \times g \text{ matrices}) = \binom{g+1}{2}.$$

Now look at the table

g	0	1	2	3	4	5	....
$\dim \mathcal{M}_g$	0	1	3	6	9	12	— — .
$\dim \mathcal{A}_g$	0	1	3	6	10	15	. — — -

Certainly, $\dim \mathcal{M}_g \geq \dim \underset{\uparrow \text{Jacobian mapping}}{\mathcal{J}}(\mathcal{M}_g)$, and so we are done. $\square$

In fact, the Jacobian mapping is an embedding — this is called the *Torelli theorem*, and we'll essentially prove it using Hodge theory later. The *Schottky problem* is to describe the Jacobian locus (image of $\mathcal{J}$) in $\mathcal{A}_g$. It is still open for $g \geq 5$.