D. Hodge decomposition

In the last section, we had

(M, h) = Hermitian manifold, $h = g - i\omega$, $\omega \in A^{1,1}_{\mathbb{R}}(M)$

$*$ = induced Hodge star, inner products $\begin{cases} \langle\,,\,\rangle_p = *((\cdot)\wedge *\overline{(\cdot)}) & \text{pointwise} \\ \langle\,,\,\rangle = \int_M (\cdot)\wedge *\overline{(\cdot)} & \text{global} \end{cases}$

Now <u>assume M Kähler</u> $(d\omega = 0)$, & set

$$L : A^k \to A^{k+2}$$

$$\Lambda := L^* = (-1)^k * L * \; : A^k \to A^{k-2}$$

(& note that these are <u>real</u> operators)

<u>Lemma 1 (Kähler identity)</u>: $[\Lambda, \bar\partial] = -i\partial^* \quad (= -i *^{-1} \bar\partial *)$

<u>Proof (sketch)</u>: Theorem I.E.4 $\Longrightarrow$ h = Euclidean metric "up to order ≥ 2"

(only 1st order differentiation of the metric occurs in the identity) $\Longrightarrow$ WMA h <u>is</u> Euclidean : $\omega = \frac{i}{2}\sum dz_j \wedge d\bar z_j$

(commutes w/ Euclidean Hodge $*$)

Writing $\bar\partial = \sum \frac{\partial}{\partial \bar z_j} \otimes (d\bar z_j \wedge)$, we see that it will suffice to check

$$(D.1) \qquad [\Lambda, d\bar z_j \wedge] = -i *^{-1}(d\bar z_j \wedge) * \qquad (\text{for each } j).$$

I'll do this up to $\pm$'s. First note

$$*(d\bar z_j \wedge) * = \pm \textcircled{2}\left(\frac{\partial}{\partial z_j} \lrcorner\right).$$

(use fact that the o.n. objects in $\langle\,,\,\rangle$ are $\{ d z_j^{(-)} / \sqrt{2} \}$)

Now $\pm$ LHS (D.1) $= *(\omega\wedge) \underbrace{*(d\bar z_j \wedge)}_{**} - \underbrace{(d\bar z_j \wedge) *}_{**}(\omega \wedge) *$

$$= \pm 2 * \{ \underbrace{(\omega\wedge)\circ\left(\tfrac{\partial}{\partial z_j}\lrcorner\right) - \left(\tfrac{\partial}{\partial z_j}\lrcorner\right)\circ(\omega\wedge)} \} *$$

$$= \pm \left(\tfrac{\partial}{\partial z_j}\lrcorner\, \omega\right)\wedge$$

$$= \pm \tfrac{i}{2}(d\bar z_j \wedge)$$

$$= \pm i * (d\bar z_j \wedge) * ,$$

done. ☐

Lemma 2: $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$.

Proof: Lemma 1 $\Rightarrow$

$$(0.2) \qquad \bar\partial\partial^* + \partial^*\bar\partial = i\big(\overbrace{\bar\partial[\Lambda,\bar\partial] + [\Lambda,\bar\partial]}^{\bar\partial\Lambda\bar\partial - \bar\partial\Lambda\bar\partial}\bar\partial\big) = 0$$

$$\Rightarrow \Delta_d = (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial)$$

$$= \Delta_\partial + \Delta_{\bar\partial} + \cancel{(\bar\partial\partial^*+\partial^*\bar\partial) + (\partial\bar\partial^*+\bar\partial^*\partial)} \to 0 \text{ (by (0.2))}$$

while

$$\Delta_\partial - \Delta_{\bar\partial} = \partial\partial^* + \partial^*\partial - \bar\partial\bar\partial^* - \bar\partial^*\bar\partial$$

conjugate of lemma 1: $[\Lambda,\partial] = i\bar\partial^*$

$$\Rightarrow i\big(\partial[\Lambda,\bar\partial] + [\Lambda,\bar\partial]\partial + \bar\partial[\Lambda,\partial] + [\Lambda,\partial]\bar\partial\big)$$

$$= i\big(\cancel{\partial\Lambda\bar\partial} - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \cancel{\bar\partial\Lambda\partial} + \cancel{\bar\partial\Lambda\partial} - \bar\partial\partial\Lambda + \Lambda\partial\bar\partial - \cancel{\partial\Lambda\bar\partial}\big)$$

note $\partial\bar\partial + \bar\partial\partial = 0$

$$= 0$$

$\square$

So we don't need to distinguish between $\mathcal{H}_d$ vs. $\mathcal{H}_{\bar\partial}$ vs. $\mathcal{H}_\partial$ in Kähler settings.

Now let M be compact Kähler.

Definition 1: $H^{p,q}(M) \left(\subset H^{p+q}_{dR}(M,\mathbb{C})\right) :=$ classes w/ representatives of type (p,q) (d-closed forms)

hence actually ∂- & $\bar\partial$-closed

One also has the "Bott-Chern" cohomology groups

$$\begin{tikzcd}
H^{p,q}_{BC}(M) := \dfrac{\{\ker d \subset A^{p,q}(M)\}}{\partial\bar\partial(A^{p-1,q-1}(M))} \arrow[r] \arrow[dr] \arrow[ddr] & H^{p,q}(M) \\
& H^{p,q}_{\bar\partial}(M) \\
& H^{p+q}(M,\mathbb{R})
\end{tikzcd}$$

w./ natural maps;

Theorem 1: There are canonical isomorphisms,*

$$\left\{\begin{aligned}
&H^k(M,\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(M), \qquad \text{(Hodge decomposition)} \\
&\overline{H^{p,q}(M)} = H^{q,p}(M), \qquad \text{and} \\
&H^{p,q}_{\bar\partial}(M) \cong H^{p,q}(M) \cong H^{p,q}_{BC}(M).
\end{aligned}\right.$$

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* written in this form, independence of the choice of Kähler metric is obvious

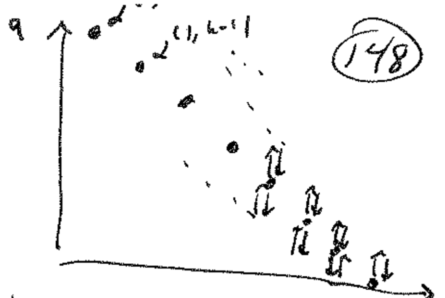

illustration of action of $\Delta_{\bar\partial}$

Proof : $\boxed{\text{Step 1}}$ Given $\alpha \in \mathcal{H}^k(M)$, write

$$\alpha = \alpha^{(k,0)} + \alpha^{(k-1,1)} + \dots + \alpha^{(0,k)} .$$

$$0 = \Delta\alpha \underset{\text{Lemma 2}}{\Longrightarrow} 0 = \Delta_{\bar\partial}\alpha$$

$$\underset{\text{diagram}}{\Longrightarrow} 0 = \Delta_{\bar\partial}\alpha^{(p,k-p)} \quad (\forall p)$$

$$\underset{\text{Lemma 2}}{\Longrightarrow} 0 = \Delta\alpha^{(p,k-p)} \quad (\forall p)$$

Hence, if $\mathcal{H}^{p,q}(M) := \mathcal{H}^k(M) \cap A^{p,q}(M)$, then

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q} \quad ; \qquad \text{and since } \Delta \text{ is real,}$$

$$\overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p} .$$

$\boxed{\text{Step 2}}$ By the Hodge theorems (+ lemma 2) we know that

$$\mathcal{H}^k \xrightarrow{\cong} H^k \ , \quad \mathcal{H}^{p,q} \xrightarrow{\cong} H^{p,q}_{\bar\partial} .$$

moreover, it is clear that

$$\mathcal{H}^{p,q} \hookrightarrow H^{p,q} \ (\subset H^k)$$

and

$$\bigoplus_{p+q=k} H^{p,q} \hookrightarrow H^k$$

(since the distinct $H^{p,q}$'s are $\perp$ under the Hodge $\langle\ ,\ \rangle$)

So

$$H^k = \mathcal{H}^k = \oplus\,\mathcal{H}^{p,q} \hookrightarrow \oplus\, H^{p,q} \hookrightarrow H^k \implies \text{inclusions are } ='\text{s},$$

and

$$\overline{H^{p,q}} = \overline{\mathcal{H}^{p,q}} = \mathcal{H}^{q,p} = H^{q,p}$$

$$H^{p,q} = \mathcal{H}^{p,q} = H^{p,q}_{\bar\partial} .$$

$\boxed{\text{Step 3}}$ To deal with $H^{p,q}_{BC}$, <u>WTS</u>: given d-closed $\omega \in A^{p,q}(M)$, <u>ω d-exact $\Rightarrow \omega$ $\partial\bar\partial$-exact</u>.

By ellipticity of the Δ's and Lemma 2,

$$(\text{D}.3) \quad \begin{cases} (\text{i}) & A^k(M) = \mathcal{H}^k \overset{\perp}{\oplus} d(A^{k-1}(M)) \overset{\perp}{\oplus} \delta(A^{k+1}(M)) \\ (\text{ii}) & A^{p,q}(M) = \mathcal{H}^{p,q} \overset{\perp}{\oplus} \bar\partial(A^{p,q-1}(M)) \overset{\perp}{\oplus} \bar\partial^*(A^{p,q+1}(M)) \\ (\text{iii}) & A^{p,q-1}(M) = \mathcal{H}^{p,q-1} \overset{\perp}{\oplus} \partial(A^{p-1,q-1}(M)) \overset{\perp}{\oplus} \partial^*(A^{p+1,q-1}(M)) . \end{cases}$$

So $\omega \in \operatorname{im}(d) \Rightarrow \omega \perp \mathcal{H}^k \Rightarrow \omega \perp \mathcal{H}^{p,q} \Rightarrow \omega = \bar\partial\eta$, and

$$\eta = h + \partial u + \partial^* v \Rightarrow \omega = \bar\partial\partial u + \bar\partial\partial^* v \overset{(\text{D}.2)}{=} -\partial\bar\partial u - \partial^*\bar\partial v$$

$$\underset{\text{apply } \partial}{\Longrightarrow} 0 = 0 - \partial(\partial^*\bar\partial v) \Longrightarrow \partial^*\bar\partial v = 0$$

$$\Longrightarrow \omega = -\partial\bar\partial u .$$

↖ recalling adjoints are injective on each others' images.

□

Now for some applications.

Definition 2 (Hodge filtration): $F^\ell A^k(M) := \sum_{\substack{p+q=k \\ p\geq \ell}} A^{p,q}(M)$

$$F^\ell H^k(M,\mathbb{C}) := \bigoplus_{\substack{p+q=k \\ p\geq \ell}} H^{p,q}(M)$$

Corollary 1: Given $\omega \in F^\ell A^k(M)$ d-closed,

(a) $[\omega] \in F^\ell H^k(M,\mathbb{C})$

(b) if $[\omega]=0$ then $\omega \in d(F^\ell A^{k-1}(M))$.

Proof: Write $\omega = \omega^{(\ell,k-\ell)} + \cdots + \omega^{(k,0)}$

Then by (D.3)(ii), $\omega^{(\ell,k-\ell)} = h^{(\ell,k-\ell)} + \bar\partial(u^{(\ell,k-\ell-1)}) + \bar\partial^*(v^{(\ell,k-\ell+1)})$

Applying $\bar\partial$ we see the last term $=0$. Replace ω by $\omega - h^{(\ell,k-\ell)} - d(u^{(\ell,k-\ell-1)})$, continue. □

Definition 3: (Betti #'s) $h^k(M) := \dim H^k(M)$

(Hodge #'s) $h^{p,q}(M) := \dim H^{p,q}(M)$

Corollary 2: $2 \mid h^{2k+1}$ $(\forall k)$

[Remark: this again shows the Hopf manifolds aren't Kähler]

Proof: $h^{2k+1} = \sum_{p+q=2k+1} h^{p,q} = 2 \sum_{\substack{p+q=2k+1 \\ p\geq k+1}} h^{p,q}$ □

Corollary 3: $\Omega^p(M) = H^{p,0}(M)$

Proof: $\omega \in \Omega^p(M) \Rightarrow \begin{cases} \bar\partial\omega = 0 & \text{(holomorphicity)} \\ \bar\partial^*\omega = 0 & \text{(type -- can't have } (p,-1)) \end{cases}$

$$\Rightarrow 0 = \Delta_{\bar\partial}\omega = \Delta\omega \Rightarrow \omega \in \mathcal{H}^{p,0}(M).$$

Conversely, $\alpha \in \mathcal{H}^{p,0}(M) \Rightarrow \alpha \in \ker \Delta_{\bar\partial} \subseteq \ker\bar\partial$. □

Ex/ $M = \left\{ \begin{pmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{pmatrix} \middle| \; x,y,z \in \mathbb{C} \right\}$

$\Gamma = M \cap GL_3(\mathbb{Z})$

Use Cor. 3 to show the Iwasawa manifold $\Gamma\backslash M$ is non-Kähler (exhibit a nonclosed holo. p-form). [Hint: Maurer-Cartan]

Definition 4: $\mathrm{Pic}(M) := H^1(M, \mathcal{O}^*)$ (gp. of line bundles)

$\mathrm{Jac}(M) := \dfrac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})}$ $= H^{0,1}$ (Jacobian torus)

$NS(M) := H^2(M,\mathbb{Z}) \cap H^{1,1}(M)$ (Neron-Severi group — more on this later)

N.B.: $H^1(M,\mathbb{Z})$ is a full lattice in $H^1(M,\mathbb{R})$, and

$$\begin{tikzcd} H^1(M,\mathbb{R}) \arrow[r, hook] \arrow[d, "\simeq"'] & H^1(M,\mathbb{C}) \arrow[d, equal, "\boxed{\text{Thm. 1}}"] \\ H^{0,1} & \underbrace{H^{1,0} \oplus H^{0,1}}_{\text{conjugate spaces}} \arrow[l, two heads] \end{tikzcd}$$

Corollary 4: $\exists$ s.e.s.

$$0 \to \mathrm{Jac}(M) \to \mathrm{Pic}(M) \to NS(M) \to 0 .$$

Proof: The exponential exact sequence (sheaves on M)

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \qquad \Longrightarrow$$

$$H^1(M,\mathbb{Z}) \xrightarrow{\alpha} H^1(M,\mathcal{O}) \to H^1(M,\mathcal{O}^*) \to H^2(M,\mathbb{Z}) \xrightarrow{\beta} \underbrace{H^2(M,\mathcal{O})}_{\substack{= H^{0,2} \\ \text{by Thm. 1}}} = \frac{H^2}{H^{1,1} \oplus H^{2,0}}$$

$\mathrm{coker}\,\alpha = \mathrm{Jac}(M)$

$\ker\beta = (H^{1,1} \oplus H^{2,0}) \cap H^2(M,\mathbb{Z}) = NS(M)$. $\square$

Example 1: It's easy to prove (say, using a cellular decomposition, or Mayer-Vietoris + induction) that

$$h^k(\mathbb{P}^n) = \begin{cases} 1, & k = 0, 2, 4, \dots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Since $h^{2l} = h^{l,l} + 2(\Sigma \cdots)$, conclude $h^{l,l}$'s $= 1$ and all other $h^{p,q} = 0$.

Upshot: $\left.\begin{array}{l} NS(\mathbb{P}^n) \cong \mathbb{Z} \\ \mathrm{Jac}(\mathbb{P}^n) \cong \{0\} \end{array}\right\} \Longrightarrow \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$, meaning that <u>the $\mathcal{O}(k)$ are all the line bundles in $\mathbb{P}^n$</u>.